UC Berkeley, Astro 218, Stellar Formation and Galactic Dynamics

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Lecture 1

- Galaxy Facts

  Stars from the most massive component (95% of mass)

  They sometimes have satellite galaxies

  Dwarf galaxies are the most common

  They come isolated or in groups

  Groups contain tens of galaxies, clusters contain thousands.

- Star Clusters vs. Galaxies

  Clusters are a few pc across; galaxies are larger than 1 kpc.

  Clusters have 10^5 stars; galaxies have over 10^7.

  Galaxies are always self-gravitating

  Galaxies contain gas, dust, and cosmic rays

- Dynamics of Stars

  The stellar mass of galaxies completely determines the dynamics.

  Calculating the collision time of stars:

$$\begin{aligned}
\ell_{mfp} &= \frac{1}{n\sigma} \\
t_{col} &= \frac{1}{n\sigma v} \\
&= \frac{1}{(.25pc^{-3})(\pi(2R_\odot)^2)(20\frac{km}{s})} \\
&\sim 2.5 \cdot 10^{18} yrs
\end{aligned}$$

  Stars never collide (collision time greater than Hubble time).

- Dynamics of Gas

Calculating collision time of gas:

$$\begin{aligned} t_{col} &= \frac{1}{n\sigma v} \\ &= \frac{1}{(1cm^{-3})(\pi(10^{11})^2)(1\frac{km}{s})} \\ &= 500yrs \end{aligned}$$

Gas collides often on timescales of galactic dynamics

- Energy Changes

  Gas energy changes via heating and cooling

  Stellar energy changes via changes in gravitational potential (ex. galaxy collisions, ejection of stars from a region)

Lecture 2

Timescale for collisional (star-star) relaxation
Suppose Star A passes Star B with impact parameter b. This will deflect the trajectory of Star A in a direction $\perp$ to the original trajectory. We'll say that a trajectory is significantly altered when $\Delta v_\perp \sim v$. Integrating over all possible b, and integrating over all pairs of stars, we will get an estimate of the timescale for star-star interactions in a galaxy.

$$F_\perp = \frac{Gm^2}{x^2+b^2}\cos\theta$$

Using $\cos\theta = \left(\frac{b}{x^2+b^2}\right)^{\frac{1}{2}}$, we have:

$$F_\perp = \frac{Gm}{b^2}\left(1+\frac{x^2}{b^2}\right)^{-\frac{3}{2}}$$

then using $F = \frac{d}{dt}p$, and $x \approx vt$, we have:

$$F_\perp = \frac{Gm}{b^2}\int_{-\infty}^{\infty}\left(1+\left(\frac{vt}{b}\right)^2\right)^{-\frac{3}{2}} dt = \Delta v_\perp$$

Substituting $s = \frac{vt}{b}$,

$$\boxed{\Delta v_\perp = \frac{2Gm}{bv}\frac{s}{(1+s^2)^{\frac{1}{2}}}\Bigg|_0^\infty = \frac{2Gm}{bv}}$$

This breaks down when $v^2 \sim \frac{2Gm}{b}$, giving us a minimum interaction distance

$$b_{min} = \frac{2Gm}{v^2}$$

For $m \sim 2 \cdot 10^{33}$, and $v \sim 2 \cdot 10^6$, b_{min} is of order 1 AU.

Note, by the way, that:

$$\Delta v_\perp = \overbrace{\frac{Gm}{b^2}}^{F_{gravity}} \overbrace{\frac{2b}{v}}^{\substack{interaction\\ timescale}}$$

Anyway, we wanted to integrate over all stars. We expect that if we throw a star through a galaxy that its net deflection $\int_* \Delta v_\perp = 0$ because stars are probably distributed symmetrically. In order to get a real measure for the interactions going on in a galaxy, we want to calculate $\Delta v_\perp^2$:

$$\sum_* \Delta v_\perp^2 dn = \left(\frac{2Gm}{bv}\right)^2 \frac{2N_{tot}}{R^2} b\ db$$

$$\Delta v_\perp^2 = \left[8\left(\frac{Gm}{Rv}\right)^2 N_{tot}\right] \int_{b_{min}}^{R} \frac{db}{b}$$
$$= \left[8\left(\frac{Gm}{Rv}\right)^2 N_{tot}\right] \ln \frac{R}{b_m in}$$

And we define $\Lambda \equiv \frac{R}{b_{min}}$. Using the definition of b_{min}, we have:

$$v^2 = \frac{GM}{R} = \frac{GmN_{tot}}{R}$$

And since $\Delta v_\perp^2 = 8v^2 \frac{\ln \Lambda}{N}$, we have

$$\frac{\Delta v_\perp^2}{v^2} = 8\frac{\ln \Lambda}{N}$$

and $\frac{\Delta v_\perp^2}{v^2} \approx 1$ represents the condition for a galaxy to have lost its "memory" of its initial conditions. Using $\Lambda = \frac{Rv^2}{2Gm}$ and $\frac{Rv^2}{m} = N_{tot}$, we have

$$\frac{\Delta v_\perp^2}{v^2} = \frac{12.5 \ln N}{N}$$

The time for a star to cross a galaxy is $t_{cross} = \frac{2R}{v}$, and the number of crossings required to relax is $n_{relax} = \frac{0.1N}{N}$. Thus, the relaxation time is

$$t_{relax} = t_{cross} n_{relax}$$
$$= \frac{0.1N}{\ln N} t_{cross}$$
$$= \frac{0.1N}{\ln N} \frac{2GM}{v^3}$$

To evaluate whether various galaxies have relaxed, we'll make a table:

The takeaway point here is that **stars in large galaxies still remember their original trajectories.**

System	$\frac{0.1N}{\ln N}$	v (km/s)	R (pc)	t cross (yrs)	t relax (yrs)	Age
Old Open Cluster: Pleideas (50)	1.28	1	2	$2 \cdot 10^6$	$1 - 2 \cdot 10^6$	$1 \cdot 10^5$?
Globular Cluster (10^6)	$7.2 \cdot 10^3$	3	3	10^6	$7 \cdot 10^9$	$1 \cdot 10^{10}$
Milky Way (10^{11})	$4 \cdot 10^8$	100	10^4	10^8	$4 \cdot 10^{16}$	$1 \cdot 10^{10}$
Dwarf Galaxy (10^7)	$6.2 \cdot 10^4$	15	$5 \cdot 10^2$	$3 \cdot 10^7$	$3 \cdot 10^{14}$	$1 \cdot 10^{10}$
Galaxy Cluster (10^3)	14	300	10^6	$3 \cdot 10^9$	$5 \cdot 10^{10}$	$1 \cdot 10^{10}$

An important number to remember is $1\frac{km}{s} = 1\frac{pc}{My}$. If a galaxy is relaxed, we may expect a thermalized velocity distribution of stars, but depending on the geometry of a galaxy, this may only apply within velocities along a particular axis.

When galaxies are ripped apart, streams of stars can be torn off into **moving groups** which can be identified by their common velocities.

Lecture 3

There is increasing evidence that galaxies form via the coalesence of satellite galaxies. We will discuss dynamical friction as a mechanism for combination. As an illustration, let's consider a point of mass M falling into a galaxy.

Let us say this galaxy is a cluster of stars of mass $m \ll M$ arranged on a flat grid. As our point mass travels through this grid, we expect the stars to fall toward it, ultimately gathering along the path behind the point mass. This will create an overdensity of stars directly behind the point mass that will serve to increase the force of gravity on the mass exiting the system relative to the force of gravity on the mass as it fell inward. If this drag is enough to keep the point mass bound to the system, we expect this process to proceed until there is an equipartition of energy between the point mass and the stars of the system.

If this point mass has a gas cloud associated with it, we'd expect this to further increase the drag of the point mass through the system as a result of shocks.

Let us formulate this mathematically. As this point mass interacts with a star at distance $\bar{r} = \bar{x}_m - \bar{x}_M$, we will see a force between them:

$$\frac{mM}{m+M}\bar{r} = -\frac{GMm}{r^2}\hat{e}_r$$

We expect the changes in the velocites of the star and the point mass that result from this interaction to be:

$$\Delta v_m - \Delta v_M = \Delta V = \bar{r}$$
$$M\Delta v_m - v\Delta V_M = 0$$
$$\Delta V - m = \frac{m}{M+m}\Delta V$$

Consider that the point mass and star pass each other at a distance (impact parameter) b. Since $m \ll M$, we will for the moment consider the point mass to be stationary, and consider the angle θ by which the path of the star deviates from a straight line. This situation is described by the Rutherford scattering equation in terms of Ψ_0, which is the angle to the "knee" of the curve describing the path of the star. Ψ in general is the parameterized angle from the scatterer to the scatteree as a function of time:

$$\frac{d^2u}{d\Psi^2} + u = \frac{G(M+m)}{L^2} = \frac{F(\frac{1}{u})}{L^2}$$

where $u \equiv \frac{1}{r}$, and $L \equiv bv_0$, where v_0 is the initial velocity of the star. This equation has the general solution of a **rosette** (a precessing elliptic curve). For our problem, this solution looks like:

$$\begin{aligned} u = \frac{1}{r} &= C\cos(\Psi - \Psi_0) + \frac{G(M+m)}{L^2} \\ \frac{d}{dt}(r) &= Cr^2\Psi\sin(\Psi - \Psi_0) \\ &= Cbv_0\sin(\Psi - \Psi_0) \end{aligned}$$

Using that $\theta = 2\Psi_0 - \pi$, $L = r^2\dot{\Psi}$, and the boundary condition that $\frac{d}{dt}r = -v_0$, we have:

$$\begin{aligned} 1 &= cb\sin(-\Psi) \\ \frac{1}{r} &= C\cos(\Psi_0) + \frac{G(m+M)}{b^2v_0^2} \end{aligned}$$

$$\boxed{\tan\Psi_0 = \frac{-bv_0^2}{G(m+M)}}$$

We may then determine what this means in terms of parallel and perpendicular velocities:

$$\begin{aligned} v_\perp &= v_0\sin\theta = v_0\,|\sin 2\Psi| = v_0\frac{|\tan\Psi_0|}{1+\tan^2\Psi_0} = \frac{v_0^3 b}{G(m+M)}\left(1 + \frac{b^2v_0^4}{G^2(m+M)}\right)^{-1} \\ v_\parallel &= v_0 - v_0\cos\theta = v_0\,|1 + \cos 2\Psi_0| = 2v_0\left[1 + \frac{b^2v^4}{G^2(m+M)^2}\right]^{-1} \end{aligned}$$

For large b, $b^2 \gg \frac{G^2(m+M)^2}{v_0^4}$, so

$$v_\perp = \frac{2Gm}{bv_0}$$

Thus we have recovered our result from last lecture for large impact parameters. We now have a result for a single encounter, so we need to examine the rate of encounters:

$$t = \frac{1}{n\sigma v}$$

This is the inverse of our rate, and $\sigma = 2\pi b\ db$. $v = v_0$, so we just need a figure for n. We will add a little formality here by introducing a density function $f(x, y, z, v_x, v_y, v_z)$ so that

$$\bar{n} = f(\bar{v}_m)d^3\bar{v}_m$$

This gives an integral for the change in the velocity of the point mass as a function of time:

$$\left.\frac{d}{dt}v_M\right|_{\bar{v}_M} = \bar{v}_0 f(\bar{v}_m)d^3\bar{v}_M\int_0^{b_{max}}\frac{2mv_0}{(M+m)}\left[1 + \frac{b^2v_0^4}{G^2(M+m)^2}\right]^{-1}2\pi b db$$

This is an integral of the form $\int \frac{k_1}{(1+k_2^2b^2)} 2\pi b db$, which we can solve, giving us the result:

$$\frac{d}{dt}v_M = \bar{v}_m f(\bar{v}_m) d^3\bar{v}_m \frac{2mv_0}{(M+m)} \frac{(M+m)^2G^2}{v_0^4} \frac{1}{2}\left(1+\frac{v^4b_{max}^2}{G^2(M+m)^2}\right)$$

Then defining $\Lambda = \frac{v_0^2 b_{max}}{G(M+m)}$ and noting that $\ln\Lambda \approx \frac{1}{2}(1+\Lambda^2)$, we have our final result:

$$\boxed{\frac{d}{dt}\bar{v}_m = -v_0\pi^2 \ln\Lambda G^2 m(M+m) \frac{\int_0^{v_M} f(v_M) v_m^2 dv_m}{v_m^3} \bar{v}_M}$$

This is the **Chandrasekhar Dynamical Friction Formula**. Often, we are only interested in the question of how long it takes a black hole to get to the bottom of a system of stars. For this purpose, we can estimate the above equation as:

$$\frac{d}{dt}v_M \approx 4\pi G^2 \frac{mM\ln\Lambda n}{v_0^2}$$

Let's examine a couple of cases. If v_m is small, then $f(v_m) \approx f(0)$. Then our integral becomes:

$$\frac{d}{dt}v_M = -\frac{16\pi^2}{3}\ln\Lambda G^2 f(0) m(M+m)\bar{v}_M$$

This says that the acceleration of a point mass is proportional to the velocity. This is the form of viscous friction (a marble in honey). On the other hand, if we say that $f(\bar{v}_m)$ is Maxwellian, then:

$$\frac{d}{dt}v_M = \frac{-4\pi G^2 \ln\Lambda(M=M)n_0 m}{v_m^3}\left[e^{4x} - \frac{2x}{\sqrt{\pi}}e^{-x^2}\right]\bar{v}_m$$

where $x \equiv \frac{v_m}{\sqrt{2}\sigma}$. Here, $\frac{\bar{v}_m}{v_0^3} \sim \frac{1}{v_0^2}$, which is similar to the frictional force of air on a bullet.

Lecture 4

The reading for next lecture is Binney & Merrifield Ch 3. Today we'll talk about **gravitational lensing** as a scattering problem. Suppose we have a particle which comes within impact parameter b of a scattering body, and is deflected by angle θ. We found last time that

$$\tan\Psi_0 = \frac{-bv_0^2}{GM}$$

where Ψ_0 is the angle to the "knee" of the path of the scattered particle. We then noticed that $\theta = 2\Psi_0 - \pi$, so that

$$\tan\left(\frac{\theta}{2}\right) = \frac{GM}{bv_0^2}$$

If this particle is a photon, and deflection angles are small (as they usually are), we can say $\tan\left(\frac{\theta}{2}\right) \approx \frac{\theta}{2}$ and $v_0 = c$, so we have:

$$\theta = \frac{2GM}{bc^2}$$

This is almost correct, and we haven't even invoked GR. If we do, the only difference we find is that of a factor of 2:

$$\boxed{\theta = \frac{4GM}{bc^2}}$$

We can plug in for our sun to find the deflection angle of light by our sun. We use $b = 7 \cdot 10^{10}$, and we find that $\theta = 8.5 \cdot 10^{-6} \approx 1.4$". This is, of course, Einstein's famous prediciton.

Let's set up a more general problem. Suppose an emitting source object (S), a lensing object (L), and an observer (O) are all in a line, so that light from the emitter can be bent around to massive object on its way to the observer. We would like to calculate the angle θ_s that light can be emitted at from the source so that it is lensed to hit the observer. We will call R_E the distance from L to the knee of the path of the light. D_{SL} is the distance from S to L, D_L is the distance from the observer to L (D_S is the distance to S), θ is the scattering angle, and θ_E is the angle at which the photon hits the observer.

Obviously, $\theta = \theta_S + \theta_E$, and for small angles, $\theta_S = \frac{R_E}{D_{SL}}$, and $\theta_E = \frac{R_E}{D_L}$. We use that $\theta = \frac{4GM}{c^2 R_E}$ and solve our original equation:

$$\boxed{R_E^2 = \frac{4GM}{c^2}\frac{D_{SL}D_L}{D_L + D_{SL}}}$$

This is an equation for the radius of an **Einstein Ring**. It is a ring, of course, because any path that passes at distance R_E around L will be deflected to O. It should be noted that a gravitational lens is not the same as a conventional lens which focusses light to a point. Rather, light that passes closer to the center is deflected more. A piece of glass which would lens light in this way would look like the bases of two wine glasses glued together along their bottom side. This is useful for developing an intuition for how light goes through a gravitational lens.
Getting back to our problem, we can solve for θ_E:

$$\theta_E = \left(\frac{4GM}{c^2}\right)^{\frac{1}{2}} \left(\frac{D_{SL}}{D_L(D_{SL} + D_L)}\right)^{\frac{1}{2}}$$

If L lies directly in the middle of S and O, then we find that:

$$\theta_E = \left(\frac{4GM}{c^2}\frac{1}{D_S}\right)^{\frac{1}{2}}$$

which is just twice the **Schwarzschild radius** divided by the twice the distance to a lensing object. We can estimate the Schwarzschild radius of the sun ($3km$) and the distance to a lensing object ($3kpc$), and we get that the angle of an Einstein ring on the sky is about 1 marcsec. This is prohibitively small for current optical telescopes, but using VLBI, radio astronomers have been able to observe these. There have only been a couple of these observed because it is very rare to find collinear S, L, and O.

Let's examine what happens if they are not collinear. Instead of getting a ring, we will get a **caustic**, which is the generalized envelope of images produced by a surface (think sun reflecting off the surface of a swimming pool). Let's consider the case that a source object (S) passes through a collection of potential scattering bodies before it reaches the observer. In order to be deflected, light must pass within an Einstein radius of one of these bodies. We can calculate the optical depth τ of the scattering bodies to lensing:

$$\tau = N\pi\theta_E^2$$

where N is the number of scattering bodies *per steradian.* In general τ is quite small. We'll estimate τ for stars in the bulge of the Milky Way.

$$N \approx \frac{\#stars}{\Omega_{bulge}}$$

Now $\#stars = \frac{M_{tot}}{\langle M_* \rangle}$, and $\langle M_H \rangle \approx 0.4 M_\odot$, giving us $\#stars \approx 2.5 \cdot 10^{10}$. The bulge on the sky is about $20° \times 30° \sim \frac{1}{6} str$. Plugging these values in, we find

$$\tau_{bulge} \sim 4.5 \cdot 10^{-6}$$

That is, we need to look at several million stars to find a lensing event. Now that we have computers looking for these events, we find them, but this would be impossible to do by hand. This is why lensing has only recently become a hot topic in research. If we repeated this calculation for the disk, we get $\tau_{disk} \sim 6 \cdot 10^{-6}$, which is about the same.

Microlensing is the effect of a lensing object increasing the number of pathways for light to hit an observer, making an object appear to brighten and then dim as a lensing object passes by. These events may be distinguished by many other events by the fact that it does not depend on wavelength.

Let's set up one more problem: that of the non-collinear S, L, and O. We'll define θ to be the angle the incoming light makes with the path from O to L, α is the angle that light is deflected from the straight line it was following out from the source, and β is the angle between the lines OL and OS. We'll also define the point I to be where O "observes" S to be (where the image is).
We know $\alpha = \frac{4GM}{c^2 b}$, and $\theta_E^2 = \frac{4GM}{c^2} \frac{D_{SL}}{D_S D_L}$. $b = D_L \theta$, and we'll define $\eta \equiv D_S \beta = D_S \theta - D_{SL} \alpha$. We then find:

$$\boxed{\beta = \theta - \frac{\theta_E^2}{\theta}}$$

The above equation is written incorrectly in Binney & Tremaine. We may solve for θ as well, using $\mu \equiv \frac{\beta}{\theta_E}$:

$$\frac{\theta}{\theta_E} = \frac{\mu \pm \sqrt{\mu^2 + 4}}{2}$$

There is also a corresponding light path which goes on the other side of L and reach O. If we cannot resolve this split, then our telescope will just detect the fact that the solid angle for that source will have increased on the sky, and since photons are conserved per solid angle, we will see that as a brightening. We may actually work out the solid angles represented on the sky on each side of L of our source S (called A_+ and A_- for the brighter (close side) and dimmer (far side) of L). Working out the arithmetic, we find:

$$\boxed{\frac{A_\pm}{A_S} = \frac{1}{2}\left(1 \pm \frac{\mu^2 + 2}{\mu(\mu^2 + 4)^{\frac{1}{2}}}\right)}$$

Lecture 5

For next time, we should be reading Binney & Merrifield Chapter 4.

When we look at galaxies, we get information about their brightness and spectrum. From this, we would like to make inferences about their evolution. To do this, we need information about the mass of the galaxy. Since most of the visible mass of a galaxy is in the form of stars, it is important to have a solid understanding of stars to make inferences about observations of galaxies.

We measure the masses of stars using binary star systems and applying Kepler's 3rd law. Ultimately, we would like to have a **stellar mass function** $\Phi(M)$ that gives us the number of stars of a given mass per unit volume. This function depends on time because of the effects of stellar evolution. The **initial mass function** $\Psi(M)$ has these effects removed. A current topic of research is how/whether $\Psi(M)$ varies over the history of the universe.

The mass of a star depends on the magnitude of a Jean's instability in a region. Since Jean's instabilities can have masses lower than that required for H-He fusion, we have, in addition to main sequence stars, **brown dwarfs** such as L stars ($T = 1500 - 2000K$) which contain Lithium, and T stars ($T < 1000K$) which have methane. These stars are sustained by deuterium fusion.

We can measure $\Phi(M_V)$, the **general luminosity function**, where M_V is the magnitude of light in the visible band. The simplest way to do this is to take a picture with a CCD, measure the parallax of each of the stars in the picture, and then plot $\Phi(M_V)$ vs. N. However, there are two biases which make this an inaccurate measurement. First, there is the Malmquist Bias, which states that since there is a limit to the sensitivity of any instrument, we can see brighter stars out further than dim stars. Thus, any picture we take will be sampling different volumes for different kinds of stars. This effect needs to be removed to determine $\Phi(M_V)$, which is supposed to be a per volume measurement.

We can do this by doing proper-motion surveys to determine which are the closer stars and determining the ratios of star types for a given distance. The second bias is the Lutz-Kelker Bias, which states that when we take a picture of the sky, we are sampling a solid angle on the sky. If we have some uncertainty in the aperature of our telescope, we will have an error in our calculated solid angle. This error will have a greater effect on measurements of far-away stars than on close-by stars. In proper-motion surveys, this will tend to cause us to oversample distant stars, and will introduce a second bias as we try to correct the first bias. There are additional problems in measuring $\Phi(M_V)$. For example, we don't even know M_V because of extinction due to interstellar dust.

We would then like to work from $\Phi(M_V)$ to $\Phi(M)$. To do this, we use theoretical stellar models which are informed by observations of binary stars. We restrict our calculations to stars on the main sequence in order to get a direct mapping of luminosity to mass. This is stellar structure theory.

Important general trends to know about the solar vicinity are: most of the light (luminosity) is from O and B stars, most stars are M stars, and most of the mass is in K and M stars.

To work from $\Phi(M)$ to $\Psi(M)$, we need to use stellar evolutionary theory (the fact that big stars die young). The first work of this type was done by Salpeter in 1954. This isn't as simple as dividing each star type by it's lifetime because $\Phi(M)$ has sampled multiple generations of stars, and so it includes the dependence of **star formation rate** on star size. This is especially problematic when observing far-away galaxies because we have no clue about how star formation rates vary between galaxies and as a function of time since the big bang.

Lecture 6

Everyone acknowledges that astronomy's magnitude system sucks. Nonetheless, it is necessary to understand it:

$$m_1 - m_2 = -2.5 \log_{10} \left(\frac{f_1}{f_2} \right)$$
$$m - M = 5 \log d - 5$$
$$m - M = 5 \log d - 5 + A + K$$

where m_1,m_2 are two apparent magnitudes. M is the standardized magnitude of a star at a fiducial distance of $10pc$. A accounts for foreground extinction from dust, and K corrects for the redshift of galaxies:

$$K = K_0 + 2.5 \log(1 + z)$$

K is important to account for. Since galaxy measurements are taken with particular waveband filters, there are all sorts brightening/dimming effects which are introduced by spectral features getting redshifted into/out of the wavebands where we are taking measurements. K is also hard to remove because it requires knowledge of K_0, which means we have to know ad hoc what the spectrum of our galaxy looks like.

Dust

To work out A, we need to first measure a **color excess** $E(B - V)$, where $E(X - Y)$ is given by:

$$E(X_{\lambda_0} - Y_{\lambda_1}) = (m(X) - m(Y)) - (m_0(X) - m_0(Y))$$

where m_0 describes the unreddened magnitude. We may then compute the extinction at X as:

$$A_X = (m - m_0)_X$$

Since reddening asymptotes to 0 at long wavelengths, we can try to figure the ratio of the total extinction to the color excess at each waveband. Emperically, this has been measured in our neighborhood, and we find:

$$\boxed{\frac{A_V}{E(B - V)} = 3.0}$$

where A_V is the magnitude of extinction in the V band (which was chosen to be the band that all color excesses are related to).

There was an interesting measurement done by Copernicus in the 1970's which related the gas mass to dust mass along a line of sight. He found:

$$\frac{M_{gas}}{M_{dust}} = 100$$

and this did not seem to change with the line of sight. This will be very useful to us because we can measure hydrogen columns quite well with 21 cm radiation, so we can get good figures for the amount of dust along a line of sight.

As long as we are measuring primarily starlight, we can keep relating the magnitudes of exinctions to A_V. For example:

$$\frac{A_V}{A_K} \sim 10$$

Just as a reference, the order of wavebands goes: U, B, V, R, I, J, K, L, M. L is about $3.5\mu m$, and redward of M, we start getting features introduced by dust, and our system of comparing extinctions relative to A_V breaks down.

Eddington Luminosity

The Eddington Luminosity is the classic point at which a star's radiation pressure balances the gravitational pull of the star on hydrogen atoms. The key constant which determines where this balance is (other than gravity), is the **Thomson Crossection**:

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 = 6.65 \cdot 10^{-25} cm^2$$

We then balance the force of gravity $F_g = \frac{d}{dt}p$ with the per-photon momentum transfer:

$$\frac{d}{dt}p = \frac{1}{c}\frac{d}{dt}E = \frac{L_*}{c}\frac{\sigma_T}{4\pi R_*^2}$$

Now $F_g = \frac{M_* m_p}{R_*^2}$, so we find:

$$\frac{L_*}{M_*} = \frac{4\pi G c m_p}{\sigma_T} = 6.31 \cdot 10^4 \frac{ergs}{sec \cdot g}$$

Besides being relevant for capping star-formation at O stars, this relevant in galactic centers. By the way, we made the assumption of spherically symmetric accretion to calculate the Eddington limit. If star formation does not proceed symmetrically (and we know it doesn't), then this value can go significantly higher.

Now returning to our discussion of mass functions, we know the total number of stars in a system is:

$$N_{TOT} = \int_0^\infty \Psi_0(M) dM$$

where $\Psi_0(M)$ is the initial mass function again. The total mass in the system is then:

$$M_{TOT} = \int_0^\infty \Psi_0(M) M dM$$

Emperically, Salpeter found $\Psi_0(M) = N_0 M^{-\alpha} dM$, with a value of $\alpha = 2.35$. This causes our integral to break because $\alpha > 1$ means we have an infinite number of stars as we approach $M \to 0$. Similarly, $\alpha > 2$ causes the total mass to diverge as $M \to 0$. This does not mean our integrals are bad, it just means we have not adequately limited the range over which $\Psi_0(M)$ can operate. Just as there is an upper mass limit introduced by the Eddington limit, there is a minimum mass, less than which stars do not form. Emperically, we have measured this to be about $0.08 M_\odot$. This causes a **turnover** in the power law Salpeter measured. Miller/Scudo measured:

$$\Psi(M) \propto \begin{cases} M^{-2.45} & M > 10 M_\odot \\ M^{-3.27} & 10_\odot > M > 1 M_\odot \\ M^{-1.83} & 1 M_\odot > M \end{cases}$$

On the other hand, Kroupa measured:

$$\Psi(M) \propto \begin{cases} M^{-4.5} & M > 1 M_\odot \\ M^{-2.2} & 1 M_\odot > M > 0.5 M_\odot \\ M^{-1.2} & 0.5 M_\odot > M \end{cases}$$

This are obviously different results, and both of these measurements were done in our own galaxy. We would love to have a consistent number we could apply to different galaxies, but we can't even work it out for our galaxy. Extrapolating to other galaxies is a tremendous leap of faith.

There are a couple of places we can look for directly measuring $\Psi_0(M)$. We look at star-forming clusters and in the infrared measure the luminosities of clumps. This has the added bonus that pre-main-sequence stars are more luminous than main-sequence stars, so we can measure out to lower masses.

One other thing about power laws: they have no inherent scale, because you're just taking ratios. Another way of saying this is that they are self-similar. This is nice because properties of one portion of the distribution apply to the whole distribution.

Lecture 7

Today Holly Maness and Stella Offner are lecturing. We will discuss:

- Morphology

  The Hubble Tuning Fork Classification (this is not a precise classification scheme).

  Ellipticals (E0-E5), notated En, where $n = 10(1 - \frac{b}{a})$ is a measure of eccentricity. Dwarf Ellipticals (dE) are less luminous and hard to observe outside the Local Group. Dwarf Spheroidals (dSph) have very low stellar densities.

  Lenticulars (S0, SB0), which are between ellipticals and spirals. Lenticular galaxies have a disk with a very large central bulge. For normal-type lenticulars, they are enumerated (S0-S3) according to increasing dust absorption. For barred lenticulars, they are enumerated according to the prominence of the bar.

Normal-Type Spirals (Sa-Sc). Subtypes are ordered according to: bulge-to-disk light ratios, tightness with which spiral arms are wound, and the degree to which spiral arms are resolved into stars and HII regions.

Barred Spirals (SBa-SBc) are subtyped the same way as normal-type galaxies, but also by **rings** and **lenses**, noting the difference between inner rings and outer rings, etc.

Irregulars. Some examples are the Magellenic Clouds. Type I Irregulars have bright knots of O and B stars. Type II have smooth images and exhibit dust lanes. Some subtyped irregular galaxies are **Starburst** galaxies (which have jets, tails, and rings, and are formed by the merger of two disk galaxies), and AGN.

- Surface Brightness Measurements

  The specific intensity of galaxies, as we showed in 201, does not depend on distance, so you'd think we can just measure the surface brightness of galaxies directly. However, there are 5 corrections we must make to this:

  Air Glow (the luminosity of our atmosphere)

  Zodiacal Light (reflected sunlight off of dust)

  Faint/Unresolved Stars

  Extra Galactic Light

  Seeing (the displacement of photons travelling through the atmosphere).

  de Vaucouleur's $R^{\frac{1}{4}}$ Law

$$m_B = x - yR^{\frac{1}{4}}$$

Where m_B is the magnitude of brightness in the B band. This gives us the intensity of a galaxy as a function of radius:

$$I(R) = Ie^{-7.67\left[\left(\frac{R}{R_e}\right)^{\frac{1}{4}}\right]}$$

where R_e is half-light radius. However, $I(R) = \int_{-\infty}^{\infty} j(r)dz$, which is to say that our observation of the surface brightness of a galaxy is actually the the integral of the emissivity of the galaxy along the z axis. Taking these we get the **Modified Hubble Law:**

$$I(R) = \frac{I_0}{1 + \left(\frac{R}{r_0}\right)^2} \leftrightarrow j(r) = \frac{j_0}{\left[1 + \left(\frac{r}{r_0}\right)^2\right]^{\frac{3}{2}}}$$

Note that R is the actual radius of a galaxy, and r is the radius at which we are viewing a galaxy. Using these equations, we may derive the total luminosity of a galaxy as a function of radius:

$$L(R) = 2\pi \int_0^R I(R)R'dR' = 2\pi \int \frac{I_0 R'}{1 + \left(\frac{R}{r_0}\right)^2} dR'$$

$$= I_{-}\pi \ln\left(1 + \left(\frac{R}{r_0}\right)^2\right) r_0^2 \Big|_0^R$$

This diverges as $R \to \infty$. To prevent this from happening, we have to model the emissivity as:

$$j(r) = \frac{3-\gamma}{4\pi}\frac{La}{r^\gamma(r+a)}$$

It is important to note that for a radially symmetric galaxy, if we observe it to be centrally concentrated in surface brightness, it must be even more highly concentrated volumetrically.

- The Galaxy Luminosity Function $\Phi(L)$

How many galaxies of each (luminosity) type are there in a representative volume of the universe? To answer this question we discuss the Galaxy Luminosity Function:

$$\Phi(L) = \frac{dn}{dL} = \frac{\#}{V \cdot L}$$

Observations are taken to solve the equation:

$$B - M_B = 5\log d - 5 + A_B + K$$

However, these observations are subject to several biases:

The Malmquist (Volume) Bias

d is poorly determinted for nearby galaxies, where peculiar velocities outweigh Hubble velocities.

Galaxies are not uniformly distributed in space.

Accounting for these biases as best we can, we find that for low luminosities, the number of galaxies goes as a power law, and then steepens to an exponential falloff at high luminosities:

$$\Phi(L) = \frac{\phi_*}{L_*}e^{-\frac{L}{L_*}}\left(\frac{L}{L_*}\right)^\alpha$$

where ϕ_* is a normalization factor for the mean density and is of order 1, and $\alpha \approx -1$. This is called the **Schechter Function**.

We could try to calculate the luminosity density of the universe:

$$\begin{aligned} L_{tot} &= \int_0^\infty \frac{dn}{dL} L dL \\ &= \phi_* \int_0^\infty \left(\frac{L}{L_*}\right)^{\alpha+1} e^{-\frac{L}{L_*}} dL \quad = \phi_* L_X \Gamma(\alpha+2) \end{aligned}$$

Where $L_* \sim 10^{10} L_\odot$. Note that the luminosity of the Milky Way is about $10^{10} L_\odot$

We then may attempt to compute the # density of Galaxies:

$$\begin{aligned} n &= \int_0^\infty \frac{dn}{dL} dL \\ &= \phi_* \underbrace{\Gamma(\alpha+1)}_{divergent} \end{aligned}$$

To prevent divergence, we can't integrate from $L = 0$.

- Global Correlations of Elliptical Galaxies

  We try to classify ellipticals by shape: (eccentricity, "boxiness", velocity dispersions).

  Also, by "nonshape" characteristics: (luminosity L_e, radius R_e, Surface-Brightness I_e, B-V, σ).

  Surface-Brightness is related to the effective radius R_e:

$$R_e \propto \langle I_e \rangle^{-0.83}$$

Then using $L_e = \pi R_e^2 \langle I_e \rangle$, we have:

$$L_e \propto I_e^{-0.66}$$

This tells us that larger galaxies have less average surface brightness, and that more luminous galaxies also have less average surface brightness.

The **Faber-Jackson** relationship says that

$$L_e \propto \sigma_0^4$$

where σ_0 is the central velocity dispersion. This tells us that larger galaxies have higher velocity dispersions.

The color-magnitude relation was the observation that more luminous galaxies have stronger absorption lines, and that more luminous galaxies are redder. Since reddening is related to metallicity and age of a galaxy (fewer blue stars), this has important implications for galaxy evolution.

Relating the three quantities $\log(R_e)$, $\langle I_e \rangle$, and σ, we have 2 independent variables. This is called the **Fundamental Plane Relation**. In the space of these 3 variables, the normal vector of this plane is $(-0.65, 0.22, 0.86)$. An edge of the plane is given by:

$$\log(R_e) = 0.36 \left(\frac{\langle I_e \rangle}{\mu_B} \right) + 1.4 \log \sigma$$

We need to come up with a model of galaxy formation which accounts for the flatness of this distribution.

Finally, we have the **Dn-sigma relation**, where D_n is the diameter at which the intensity equals $20.75\mu_B$. Using the $R^{\frac{1}{4}}$ law and the fundamental plane relation, we find that

$$\frac{\sigma_0^{\frac{4}{3}}}{D_n} \propto const$$

Measurements of the Virgo Cluster have determined that:

$$\frac{D_n}{kpc} = 2.05 \left(\frac{\sigma}{100 \frac{km}{s}} \right)^{1.33}$$

This relation is good to about a factor of 2 because of inherent scatter in this relation for different elliptical galaxies.

Lecture 8

The **Tully-Fisher** relationship says that if we plot the luminosity of a (spiral) galaxy as a function of the circular velocity ($v_c R$), which is the (constant) rotational velocity of a galaxy with effects of projection removed by assuming the galaxy is circular, we find that $\log(L)$ grows linearly with v_c, and that morphologically, increasing luminosities go like S_c, S_b, S_a, S_0. This is very useful because it tells us the inherent luminosity of a galaxy based on measurements we can take. This tells us the distance to a galaxy, has can be used for cosmology. This relationship can be tightened by removing the effects of dust extinction–by measuring in infrared. Doing this, the relationship tightens to instrumental error. Therefore, something fundamental is being indicated in this relationship.

Looking at slides about the Local Group, we find that around the Milky Way and Andromeda, are clustered a bunch of dwarf spheroidals. We then look at a plot of galaxies outside the Local Group. Some of them aren't so far away, but they aren't included in the Local Group. This is because, measuring their velocities, we find that they are not bound gravitationally to us. When we measure velocity of M31 (Andromeda), it is approaching us. By the way, Andromeda outweighs the Milky Way, but they together account for 90% of the mass of the Local Group. **We define the Local Group to be all galaxies within 1200 kpc, which is the distance to the surface of 0 velocity where infall balances Hubble recession.**

Historically, we've known of the "missing mass" problem since Zwicky measured the luminous mass of galaxies, and compared that to the gravitational mass required for stable rotation. The confirmation of this problem came when we try to account for the infall of the Milky Way and Andromeda by the luminous mass of each. Even lower bounds which assume radial infall and this being the first approach of the two galaxies give mass estimates which are an order of magnitude above what we observe.

The **Seeing** problem is the problem of the Airy disc (beam) of the telescope being scattered as a result of the index of refraction through the atmosphere changing along different paths as a result of density disturbances caused by turbulence. This turbulence has a power law, and is a problem on all scales.

Lecture 9

We will have a midterm October 20th.

Getting back to what a galaxy is, we look at M100, which is a spectacular spiral galaxy. It is important to note that between the spiral arms of a galaxy, even though we don't see much brightness, there are as many stars as in the spiral arms. The spiral arms are bright as a result of the initial luminosity function: OB stars are alive and very bright.

We then look at M33, which has less well defined spiral arms when viewed in the continuum, but pronounced arms are visible when a technique called **unsharp masking** is used to process the image. This technique subtracts a "jiggled" image from the original image to undo the effects of isophotometric rings tricking our eyes away from the spirals.

In the text, there are some spectacular pictures of rings in elliptical galaxies. This is the result of **spindown** caused by dynamical friction.

Potential Theory

So far, we've been discussing galaxies as collections of point-masses. In this model, we have to do N-body simulations to understand the dynamics of galaxies. However, we can make some simplifying assumptions. If we consider the gravitational influence of a region on a point, we can describe the force as:

$$F(\vec{x}) = G\int \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3}\rho(\vec{x}')d^3\vec{x}'$$

$$\Phi(\vec{x}) = -G\int \frac{\rho\vec{x}'}{|\vec{x}' - \vec{x}|}d^3\vec{x}'$$

$$\vec{\nabla}_k\left(\frac{1}{|\vec{x}' - \vec{x}|}\right) = \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3}$$

$$F(\vec{x}) = \vec{\nabla}_x\int \frac{G\rho(\vec{x}')}{|\vec{x}' - \vec{x}|}d^3\vec{x}$$

$$F(\vec{x}) = -\vec{\nabla}\Phi(\vec{x})$$

$$\int_V \vec{\nabla}\cdot\vec{F}d^3x = \int_S \vec{F}\cdot dA$$

$$\vec{\nabla}\cdot\vec{F} = -4\pi G\rho(x)$$

$$\nabla^2\Phi = -4\pi G\rho$$

Thus, we have defined a potential field Φ (potential energy per mass), and we can get our energy from this field as:

$$W = \frac{1}{2}\int \rho(x)\Phi(x)d^3x$$

Potential fields are nice because they are scalar fields. If we say that our total potential is the sum of various components:

$$\Phi_{TOT} = \Phi_x + \Phi_y + \Phi_z$$

then we can determine our potential as a function of radius for circularly orbiting matter:

$$\frac{d\Phi}{dR} = \frac{v_x^2}{R} + \frac{v_y^2}{R} + \frac{v_z^2}{R}$$

We know that for a centrally concentrated object, its gravitational potential is $\Phi = \frac{GM}{R}$. We may then define **circular velocity** to be:

$$\boxed{v_c^2 = R\frac{d\Phi}{dR}}$$

where $v_c^2 = v_x^2 + v_y^2 + v_z^2$. We can also define **escape velocity** to be:

$$v_{esc}^2 = 2\left|\Phi(R)\right|$$

Note that this tells us that the units of potential are *velocity*^{2}. We go on to prove that for a spherically symmetric density function, all shells exterior to a point do not exert a force on that object (although they *do* create a constant, non-zero potential), and all shells interior behave as if all their mass were concentrated at the center. If we do not have a spherically symmetric density function, then we may break up the integral over ρ as a function of r as:

$$\boxed{\Phi = -4\pi G \left[\frac{1}{r} \int_0^r r^2 \rho(r) dr + \int_r^\infty r\rho(r) dr \right]}$$

We can verify from this equation that for an object with diameter $a < r$, that $\Phi = \frac{GM}{r}$. A curiousity is that if Φ is constant in a sphere, then $v_c = r\frac{d\Phi}{dr} = \left(\frac{4}{3}\pi G\rho\right)^{\frac{1}{2}} r$. Then the orbital period:

$$T = \frac{2\pi r}{v_c} = \left(\frac{3\pi}{G\rho}\right)^{\frac{1}{2}}$$

does not depend on radius. This happens in nature: in galaxies we find that we have a density function that turns over. Therefore, for some region in a galaxy, there is a corotating belt.

The time it takes a particle to fall to the center from a radius r is $\frac{T}{4} = \left(\frac{3}{16\pi G\rho}\right)^{\frac{1}{2}} = t_{dyn}$ and is called the **dynamical time**. The dynamical time is not quite the same as the free-fall time, because free-fall time assumes the mass interior to a particle is constant, whereas the dynamical time assumes constant density. The two are related: $t_{ff} = \frac{t_{dyn}}{\sqrt{2}}$.

Ultimately, what we are going for here is to estimate mass distributions in spiral galaxies. We'll assume that $v_{gas} = v_c$ because gas cannot support crossing orbits (the gas would then shock until it is all flowing together circularly). Thus, we can measure the circular velocity (and thus, Φ), and figure this out.

Lecture 10

For next time, we'll need to read Binney & Merrifield Ch. 8 (pg. 451-487). A question that arose when Hubble began classifying elliptical galaxies by their eccentricity is whether the observed "flattening" is the result of the galaxy's rotation. We can characterize this by comparing the energy of rotation to the energy of gravitation:

$$\begin{aligned} E_{rot} &= E_{grav} \\ \frac{1}{2} I w^2 &= \frac{GM^2}{R} \\ \frac{1}{2} M R^2 \frac{v_c^2}{R^2} &= \frac{1}{2} M \sigma_v^2 \\ \frac{v_c}{\sigma_v} &= 1 \end{aligned}$$

Experimentally, we have recently measured $\boxed{\frac{v_c}{\sigma_v} < 1}$, so we know that **elliptical galaxies are not rotationally supported**. This calculation was just for stars. If we were to do this for gas, we would have to include the effects of pressure. However, most galaxies are not pressure supported, so this result should hold. Spiral galaxies *are* rotationally supported, with a ratio of about 10.

Now we will talk about two kinds of systems: those with **power law density distributions** and **disks**.

Systems with Power Law Density Distributions

We'll say that the density is given by:

$$\rho(r) = \rho_0 \left(\frac{r_0}{r}\right)^\alpha$$

We would like to derive expressions for the surface density $\Sigma(R)$, v_c, and $\Phi(R)$. R is the distance from the center along the *surface* that we see. If x is the vertical distance along the line of sight to the center of a galaxy, we can say:

$$dx \cos\phi = r d\phi$$

$$\frac{x}{R} = \tan\phi$$

$$R = r\cos\phi$$

$$x = r\sin\phi dx = r\frac{d\phi}{\cos\phi}$$

Now $r = (x^2 + R^2)^{\frac{1}{2}}$, so $\phi(x) = \phi_0 r_0^\alpha (x^2 + R^2)^{-\frac{\alpha}{2}}$. We then have:

$$\Sigma(R) = 2\int_0^\infty \rho(x) dx \Sigma(R) = 2\rho_0 r_0^\alpha \int_0^{\frac{\pi}{2}} \frac{R^{(1-\alpha)}}{\cos^(1-\alpha)\phi} \frac{d\phi}{\cos\phi}$$

This yields:

$$\boxed{\Sigma(R) = \frac{2\rho_0 r_0^\alpha}{R^{\alpha-1}} \int_0^{\frac{\pi}{2}} \cos^{(\alpha-2)} \phi d\phi}$$

For the special case that $\alpha = 2$, we have:

$$\Sigma(R) = \frac{2\rho r_0^\alpha}{R^{\alpha-1}} \int_0^{\frac{\pi}{2} d\phi}$$

$$\boxed{\Sigma(R) = \frac{\pi r_0^2 \rho_0}{R}}$$

This is called a **singular isothermal sphere**. In the more general case:

$$\int_0^{\frac{\pi}{2}} \cos^h \phi d\phi = \frac{\frac{\sqrt{\pi}}{2}\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}$$

$$\Sigma(R) = \frac{\rho_0 r_0^\alpha}{R^{\alpha-1}} \frac{(-\frac{1}{2})!\left(\frac{\alpha-3}{2}\right)!}{\left(\frac{\alpha-2}{2}\right)!}$$

$$M(R) = \frac{4\pi}{3-\alpha}\rho_0 r_0^\alpha r^{3-\alpha}$$

$$v_c^2(R) = \frac{4\pi}{(3-\alpha)} G\rho_0 r_0^\alpha r^(2-\alpha)$$

Note that for a singular isothermal sphere, $v_c(R)$ is constant.

$$v_e^2(r')2\int_0^\infty \frac{GM(r')}{r'^2}dr' = \frac{8\pi G\rho_0 r_0^\alpha}{(3-\alpha)(\alpha-2)}r^{2-\alpha}$$
$$= \frac{2v_c^2}{\alpha=2}$$

This requires $\alpha > 2$. Thus, we have:

$$\left(\frac{v_e}{v_c}\right)^2 = \frac{2}{\alpha-2}$$

Disks

For disks, we need to write a **Poisson Equation** in cylindrical coordinates.

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{d\Phi}{dR}\right) + \frac{\partial^2\Phi}{\partial z^2} = 4\pi G\rho$$

Using seperation of variables, we'll say $\Phi(R,z) = J(R)Z(z)$. Then

$$\frac{1}{J(R)}\frac{d}{dR}\left(R\frac{dJ}{dR}\right) = -\frac{1}{Z(z)}\frac{d^2Z}{dz^2}$$

This last looks like a harmonic oscillator equation with constant $k = \sqrt{\frac{1}{J(R)}\frac{d}{dR}\left(R\frac{dJ}{dR}\right)}$:

$$\frac{d^2Z}{dz^2} = -k^2Z(z)$$
$$Z(z) = Se^{\pm kz}$$

We can then work out that:

$$\Phi(R,z) = e^{\pm kz}J_0(kR)$$

We can read more about this at (Toomre, 1963, ApJ 138, 385), and about disks at (Freeman, 1970, ApJ 160, 811). From observations, we can say:

$$\Sigma(R) = \Sigma_0 e^{-\left(\frac{R}{R_d}\right)}$$
$$\Phi(R,z) = -2\pi G\Sigma_0 R_d^2\int_0^\infty \frac{J_0(kR)e^{-k|z|}}{(1+(kR_d)^2)^{\frac{3}{2}}}dk$$
$$\Phi(R,0) = -4\pi G\Sigma_0 R\left[I_0\left(\frac{R}{R_d}\right)K_1\left(\frac{R}{R_d}\right) - I_1\left(\frac{R}{R_d}\right)K_0\left(\frac{R}{R_d}\right)\right]$$
$$v_c^2(R) = R\frac{d\phi}{dR} = 4\pi G\Sigma_0 R_d\left(\frac{R}{R_d}\right)^2\left[I_0\left(\frac{R}{R_d}\right)K_0\left(\frac{R}{R_d}\right) - I_1\left(\frac{R}{R_d}\right)K_1\left(\frac{R}{R_d}\right)\right]$$

Where K_n's are the **modified Bessel Functions**.

$$I_\nu(\left(\frac{R}{R_d}\right)) = e^{\frac{-i\pi\nu}{2}}J_\nu(i\left(\frac{R}{R_d}\right))$$
$$K_\nu(\left(\frac{R}{R_d}\right)) = \lim_{\nu\to 0}\frac{\pi}{2}\frac{I_{-\nu}(\left(\frac{R}{R_d}\right)) - I_\nu(\left(\frac{R}{R_d}\right))}{\sin\nu\left(\frac{R}{R_d}\right)}$$

Thus we have:

$$I_0(\left(\frac{R}{R_d}\right)) = J_0(i\left(\frac{R}{R_d}\right))$$
$$I_1(\left(\frac{R}{R_d}\right)) = -iJ_1(i\left(\frac{R}{R_d}\right))$$
$$I_{-1}(\left(\frac{R}{R_d}\right)) = iJ_1(i\left(\frac{R}{R_d}\right))$$

To remind us of what these modified Bessel functions are:

$$\begin{aligned} K_0(x) &\approx -\ln(x) \\ K_1(x) &= \frac{\pi}{2}\frac{iJ_1(ix) + iJ_1(ix)}{\sin x} \\ &= \pi\frac{J_1(ix)}{\sin x} \\ J_\nu(x) &\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(\nu+k)!}\left(\frac{1}{2}x\right)^{\nu+2k} \end{aligned}$$

For large x, $I_\nu(x) \to \frac{e^2}{\sqrt{2\pi x}}$ and $K_\nu(x) \to \sqrt{\frac{\pi}{2x}}e^{-x}$.
For small x, $I_\nu(x) \to \frac{1}{x!}\left(\frac{1}{2}x\right)^\nu$, and $K_\nu(x) \to \frac{(\nu-1)!}{2}\left(\frac{1}{2}x\right) - \nu$.
Thus, $J_\nu(x) = \sqrt{\frac{2}{\pi x}}\cos(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)$.
Plugging this into the equations we've derived, we find that as $\left(\frac{R}{R_d}\right) \to 0$, a near-zero approximation gives us:

$$\boxed{v_c^2(R) = 4\pi G\Sigma_0 R_d\left(\left(\frac{R}{R_d}\right)\right)^2\left[\ln\frac{2R_d}{R} - \frac{1}{2}\right]}$$

We can graph this, but it looks vaguely like a random distribution with a left bound. We have now developed the tools we need to be able to compute $\frac{v_c^2}{R}$ for the sphere and the disk, and since $\Phi_{TOT} = \Phi_1 + \Phi_2 + ...$, we can compute the total v_c of most galaxies. We can plot v_c versus radius and observe that up to R_c, v_c is dominated by the bulge, between R_c and $2R_d$ we see a superposition of the two influences, and beyond $2R_d$, v_c is dominated by the disk.

$$v_c^2(r) = \frac{4\pi}{(3-\alpha)}G\rho_0 r_0^\alpha r^{2-\alpha}$$

However, this predicts that as r increases, $v_c \to 0$. We know this is not the case: in fact, $v_c \to$ *constant*. This tells us there must be some matter there (which we are not seeing), that is behaving like an isothermal sphere–that is, it must have a $\frac{1}{r}$ density profile.

Lecture 11

Today we'll talk about the measurement of rotation curves. This will cover Chapter 8 of Binney & Merrifield. Rotation curves are usually measured using the 21 cm line. As before, we note that we have approximately linear growth in $V(r)$ as a function of radius, until we hit a regime where

$V(r) \sim constant$. For linearly growing $V(r) = Kr$, we derived that $\rho = constant$.

It is important to not that when we measure rotational curves of galaxies, we are almost always measuring the velocity of gas. This gas comes in 3 forms:

- Atomic Gas

  Characterized by electronic transitions in optical/UV wavebands. To excite these transitions, gas must be hot.

  We also see 21 cm radiation. We can estimate the inherent width of the 21 cm line by saying $\Delta E \Delta t \geq \hbar$. Using $\Delta E = h\Delta\nu$, and $\Delta t = 3 \cdot 10^{14} s$. Plugging this in, we find that $\Delta\nu \sim 5 \cdot 10^{-16} Hz$, so we can ignore the inherent line width of the 21 cm line. However, we cannot ignore the effects of temperature on linewidth. For high-end temperatures $T \sim 10^4$, we get $v \sim 10\frac{km}{s}$. The lowest T we expect to see is $3K$, which gives us $v \sim 0.17\frac{km}{s}$. This sets a lower bound on the resolution needed for 21 cm spectrometers.

- Molecular Gas

  The most common molecule in the ISM is H_2 (99.98%). H_2 has no dipole transition ($J = 1 \rightarrow 0$), so the strongest rotational transition in H_2 is $J = 2 \rightarrow 0$ at $28\mu m$, and this is a weak (forbidden) transition.

  Since H_2 is hard to detect, we look more to other abundant molecules: CO, CNO, H_20, NH_3. These molecules have dipole transitions, and are much easier to see. Most molecules form in high-density regions. The critical density for populating the first rotational transition of CO is $n \sim 10^3 cm^{-3}$. However, we typically observe the J_{10} transition of CO to be thermalized around $n \sim 0.1 cm^{-3}$. This is caused by collisions with H_2. We use CO as a tracer for H_2, because it forms at comparable densities, and is dissociated at comparable temperatures.

- Ionized Gas

  One region of pressure equilibrium in the ISM are $10^4 K$ HII regions, typically around O stars. Around O7 stars, there are stable regions of $10^6 K$ gas. We'll talk more about these regions later.

In the region of a galaxy, we can typically measure I, and $\bar{v}$. We've discussed isophotes (relating to I), but now we'd like to know what lines of constant velocity look like (isovelocity contours). If a galaxy is tilted with respect to our vantage point, we'll see circles as ellipses:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and we define $\sin i \equiv \frac{b}{a}$. If R is the actual radius of the galaxy, and $\theta(R)$ is the actual rotational velocity, we'd like to determine the velocity v_r that we would measure. As discussed previously, $\theta(R) \approx constant = \theta_0$. Then:

$$v_r = \theta_0 \cos\phi \sin i$$

where ϕ is the angle around the galaxy. If v_r is constant (isovelocity contours), then $\cos\phi$ must be constant. Thus, isovelocity contours are lines through the center of the galaxy.

For next class, we should answer the question for $\theta(R) = kR$ (solid-body rotation).

Lecture 12

For solid-body rotation ($\theta(R) = kR$), v_r contours will simply be vertical lines for a galaxy viewed edge-on, with major axis along the x-axis. This approximates the actual rotation curve of a galaxy near its center. We discussed in the last lecture how galaxies at larger radii approximate $\theta(R) = k$, with radial isovelocity contours. Thus, we may draw the velocity contours as vertical near the center of a galaxy and radial farther away. In reality, this isn't far off–if you just blend the two together you get lines which look the the diverging lines of a magnetic dipole, and this is pretty close to what we measure. In fact, if $\theta(R)$ decreases at large radii, we may even see the closure of the field (i.e. the wires of coil magnet). All these types of diagrams together are generally called **spider diagrams**.

Analyzing spider diagrams can tell us a lot about the motion of stars in a galaxy. A general technique is to decompose the velocities we measure into components representing the systemic velocity (the recessional velocity of the galaxy), angular velocity (purely rotational), and infall velocity:

$$v_{tot} = v_{sys} + v_\phi \cos\phi + v_{in} \sin\phi$$

We then decompose these components as a Fourier series:

$$v_{tot} = \sum_{m=0}^{\infty} v_m + v_{m\phi} \cos(m\phi) + v_{mr} \sin(m\theta)$$

These components can then reveal circular motion, radial motion, warps, elliptical orbits, and spiral arms.

Galaxies consist of:

- stars (bulge, disk)
- HI
- H_2
- dust ($\frac{1}{100}$ of gas, by mass)

We can use v to get a figure for the mass of a galaxy ($M(R) = \frac{Rv^2}{G}$). We've mentioned already that there's a problem of "missing matter" in galaxies. To help illustrate this, we can construct a **maximum model** which contains the maximum amount allowed (by $\frac{M}{L}$) of each component in order to try and construct a model which recreates the rotation curves we measure. Gas components peak in velocity at small radii, and fall off slowly. However, they move rather slowly. Bulge components peak at slightly higher radii and fall off quickly. Finally, disk components peak at higher radii and at higher velocities, and fall off more slowly than the bulge, but more quickly than gas. The sum of all these components can create the initial flatness we see, but not out to large radii. For this we need dark matter, which must have a density profile:

$$\rho_{dark}(r) = \rho_0 \left(\frac{r}{r_0}\right)^{-2}$$

We observe a lack of matter on the local scale (50-100 kpc), the local group scale (500 kpc), and on the scale of galaxy clusters (2 Mpc). We assume for simplicity's sake that they are all the same phenomenon. We'll take a stab at guessing what dark matter is:

- Faint Stars: deep exposures of local group show no large excesses. Not likely on large scales.
- Atomic Hydrogen: DM has $M \sim 10^{12} M_\odot$ in 100 kpc. This column density of hydrogen would be easily detectable by 21 cm emission. It has not been observed.
- Molecular Hydrogen: would be detectable by CO tracer, or if there's no CO, its absorption would be seen in quasar spectra. It's not there.
- Molecular Hydrogen Iceballs: condensed H_2 couldn't be detected as long as it never passes through the disk. We can't rule this out.
- Dust: not enough primordial metallicity.
- Hot Ionized Gas: so hot we don't see optical emission lines. From Fluids, remember that in hydrostatic equilibrium:

  $$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho$$

  Using the ideal gas equation $P = \frac{\rho k T}{\mu m_H}$, we have:

  $$\begin{aligned}\frac{k}{m_H}\left(T\frac{d\rho}{dr} + \rho\frac{dT}{dr}\right) &= -\frac{GM\rho}{r^2} \\ \frac{Trk\rho}{\mu m_H}\left(\frac{1}{\rho}\frac{d\rho}{dr} + \frac{1}{T}\frac{dT}{dr}\right) &= -\frac{GM\rho}{r^2} \\ \frac{Trk\rho}{\mu m_H G}\left(\frac{d\ln(\rho)}{dr} + \frac{d\ln(T)}{dr}\right) &= M(R)\end{aligned}$$

  Using the viral temperature $\frac{1}{2}mv^2 = \frac{GMm}{r^2} = \frac{3}{2}kT$, so that $T = \frac{2GM}{3rk}$, and saying $M = 10^{12} M_\odot$ and $r = 50kpc$, we have $T = 7 \cdot 10^6 K$, which says our hydrogen would emit in x-rays. We can measure the temperature of Hot Ionized Gas with x-rays and use the gravitationally inferred total mass to get an estimate of ρ. We find that there isn't enough around galaxies to account for dark matter, but around clusters **hot gas accounts for a significant fraction of dark matter**.

Across the universe, the ratio of baryonic to dark matter is 12%.

Lecture 13

Now we're going to back up a little bit and talk some more about the components of galaxies. We start with stellar evolution.

- Main Sequence

  H→He via pp chain or CNO cycle

  Convection mixes H and He.

  He is heavier and sinks to center

  Steep T gradients need convection to get heat out; smaller gradients use radiative transport.

- HR diagram

  He flash only happens in low mass stars when He core becomes degenerate before He can ignite.

  Pre-MS branch (Hiyashi strip) varies L and R to maintain constant T.

  $M_V \left(\begin{smallmatrix} horizontal \\ branch \end{smallmatrix}\right) = 0.17\left(\frac{Fe}{H}\right) + 0.82$, so higher metallicity means dimmer stars.

  Tip of the Red Giant Branch (RGB) changes little with stellar type, so it can indicate an approximate distance.

- Initial Metallicities

  Set via Big Bang nucleosynthesis theory.

  Observationally, a star hot enought ot excite He will also create He.

  Using young, low-mass stars, we have measured $Y = \frac{He}{H} = 0.228 \pm 0.005$, and $\frac{dM_{bol}}{dY} = 3$ (bolometric magnitude = integrated over frequency) for fixed z, T (metallicity temperature).

  We can measure Y in the ISM via radio recombination lines: $n = 110 \rightarrow n = 108$ in H and He^+.

- MS lifetimes

  $t_{life} = \frac{\alpha mc^2}{L}$, where α is a fudge factor.

  $H \rightarrow He$ takes $10Gyr\frac{\frac{M}{M_\odot}}{\frac{L}{L_\odot}}$.

  $H \rightarrow Fe$ takes $0.18Gyr\frac{1}{\frac{L}{1000L_\odot}}$. This is independent of mass because stars only burn up to the Chandrasekhar limit.

  Horizontal Branch (He burning) takes 0.1 Gyr.

- Instability Strip

  κ (opacity) instability sets a vertical strip of instability in HR diagram, which covers a wider range of temperatures at higher luminosities.

  Normally, as temperature increases, opacity decreases, allowing the extra heat out.

  Near the ionization limit, opacity increases with temperature, causing heating, which builds up pressure and expands the star.

  The period of this expansion is roughly $\frac{r_*}{c_s}$.

  $P\left(\frac{\rho}{\rho_0}\right)^{\frac{1}{2}} = Q \sim \frac{1}{\sqrt{G\rho_0}} \sim 1hr$, where Q is a constant.

  The fundamental mode of expansion is lowest mode: the whole star expands. However, there are harmonics to radial expansion, and interior zero points, so that only the outer layers of the star expand.

  Solar Oscillations are not large scale, and are sensitive to conditions inside the star.

  Cepheids have both fundamental and overtone oscillations. The period-luminosity rlation is tight in K-band, but wider at B, V.

- Nuclear Physics

  There are more neutrons in heavier elements

  ^{56}Fe is the most stable, but can get heavier elements with fusion as well.

  $\alpha + \alpha$ makes it easiest to get an even mass-number.

  neutron capture in high-mass stars give other elements.

  S process (slow) absorbs n^0 and then β-decays into the next element up

  R process (rapid) absorbs lots of n^0 and the β-decays.

  the number of elements depends on initial metallicity, neutron cross-section, and radioactivity.

  Heavy elements are dispersed into the ISM via stellar winds, PNe, SN (Type II, also Ib, Ic) core collapse, and SN Type Ia

- Population Evolution

  Can use known ages of stars to study population ages and histories.

  Age and metallicity can be indicators of the historical star formation rate.

  In the Local Group:

  dSp, dE have no evidence of recent star formation; old populations

  Carina has 3 episodes of star formation; how was the gas replentished?

  Irr have on-going star formation and young populations; how is gas maintained when SN winds eject it?

Lecture 15 (14 was skipped)

The Sunyaev-Zeldovich Effect

We now discuss the **Sunyaev-Zeldovich effect**. This is the effect that a galaxy cluster has a collective mass (typically $\sim 10^{12}M_\odot$) which creates a potential well that heats (approximately virially) a proton/electron gas (for 10 Mpc, $T \sim 10^7 \to 10^8$). This hot gas interacts with the 3K (1 mm) CMB photons, which undergo **inverse Compton scattering**. This adds some energy to the CMB photons, effectively creating a copy of the CMB blackbody spectrum which is shifted slightly to higher frequencies. There is a single point at which the two blackbody spectra (the fundamental and the upscattered) overlap. This should be around 218 GHz, but if the cluster as a whole is moving towards or away from us, there will be a net red or blue shift (called the **kinematic Sunyaev-Zeldovic effect**).

How big is the SZ effect? We can measure the temperature of a cluster with ionized plasma by measuring **Bremmstrahlung**. The emission measure of this radiation is given by:

$$EM = K \int_{-R}^{R} n_e^2 dr$$

where R is the radius of the cluster, and K is some constant which may be temperature-dependent. Absorption from the SZ effect, however, only goes as n_e, and is distance-independent:

$$SZ = K \int n_e dr$$

This is because absorption only depends on the number of electrons along the line of site. Thus, we may see the SZ effect long after the x-ray bremmstrahlung flux has been diffused beyond detectability.

By measuring EM and SZ, we have 2 equations for n_e, dr, and so if we approximate that these clusters are spherical (so that $2R = \theta d$), we have the ability to directly determining the distance to these clusters. This allows us to measure Hubble's constant via the SZ effect. The optical depth to the SZ effect is given by:

$$\tau_{SZ} = \sigma_T \int dr n_e(r)$$

and the emissivity is:

$$\begin{aligned}
\epsilon(v) &= A n_e^2 T_x^{-\frac{1}{2}} e^{-\frac{h\nu}{kT_x}} \\
\epsilon(v) &= 5.44 \cdot 10^{-52} \bar{z}^2 n_e^2 T^{-\frac{1}{2}} g e^{-\frac{h\nu}{kT}} \\
\tau_{SZ} &= 2\sigma_T r_c \bar{n}_e \\
f(v) &= \frac{4}{3}\pi r_c^3 \frac{\epsilon(v)}{4\pi D^2} \\
P &= \frac{\rho k T}{\mu m_H} \\
\frac{dP}{dr} &= \frac{-GM(r)}{r^2}\rho
\end{aligned}$$

We did this last class, so we'll just cut to the chase:

$$\boxed{\frac{Trk}{\mu m_H G}\left(\frac{-d\ln\rho}{d\ln r} - \frac{d\ln T}{d\ln r}\right) = M(r)}$$

Chemical Evolution in Galaxies

Suppose we take a box with a certain mass of gas M_g. To start, we'll say this gas has metallicity $Z = 0$. Some mass of gas will get converted into stars (M_s). We'll choose a parameter p which describes the yield, or fractional increase in metallicity from an episode of star formation. We'll call $d'M_s$ the mass of new stars produced, and dM_s the mass of stars after massive stellar evolution. We define metallicity to be $Z = \frac{M_h}{M_g}$, where M_h is the mass of everything heavier than helium. If no mass leaves or enters the box, $dM_x = -dM_g$. Also, pdM_s is the mass of heavy elements produced, and so $dM_h = pdM_s$. Then:

$$\begin{aligned}
dZ &= d\left(\frac{dM_h}{dM_g}\right) \\
&= \frac{(p-Z)dM_s}{M_g} - dM_s \\
&= \frac{(p-Z)dM_s}{M_g} - \frac{x}{M_g}dM_g \\
&= -p\frac{dM_g}{M_g} \\
Z(t) &= -\frac{p\ln(M_g(t)}{M_g(0)}
\end{aligned}$$

We can then solve for the mass of gas and stars as a function of time:

$$\begin{aligned}
M_g(t) &= M_g(0)e^{\frac{-Z(t)}{p}} \\
M_s(t) &= M_t(0)\left(1 - e^{-\frac{Z(t)}{p}}\right)
\end{aligned}$$

Now let's choose α to be the fraction of the present day metallicity. It tends to be around $\frac{1}{3}$.

$$\frac{M_s[Z < \alpha Z(t)]}{M_s[Z < Z(t)]} = \frac{(1 - e^{\frac{-\alpha Z}{p}})}{(1 - e^{\frac{-Z}{p}})}$$

Now we have from before $\frac{Z}{p} = \ln\frac{M_g(t)}{M_g(0)}$, so

$$\boxed{\frac{Z}{p} = \frac{1 - \left(\frac{M_g(t)}{M_g(0)}\right)^{\alpha}}{1 - \frac{M_g(t)}{M_g(0)}}}$$

$\frac{M_g(t)}{M_t(0)} \approx 0.1$ is something we can get directly from observations (a tenth of the total luminous mass of the galaxy is in gas). For stars:

$$\frac{M_s(\alpha Z(t))}{M_s(Z(t))} = \frac{(1-0.1)^{\frac{1}{2}}}{(1-0.1)} = 0.6$$

However, we measure this fraction to be much smaller. This is called the **G-dwarf problem**. Historically, it was thought that G-dwarfs might have been a solution to this. Now we know that it isn't, and so we need to talk about accreting/leaky boxes.

Lecture 16

- Leaky Box Model

  Gas leaves system via galactic winds from SNe

$$\frac{d}{dt}M_{tot} = -C\frac{d}{dt}M_s$$

Where M_s is the mass in stars, and M_{tot} is the mass in stars and gas.

Going through the same reasoning as before, we find:

$$M_g(t) = \frac{M_g(0)}{1+C} e^{-\frac{(1+C)Z}{\rho}}$$
$$M_s(t) = \frac{M_{tot}(0)}{1+C} e^{-\frac{(1+C)Z}{\rho}}$$

This can explain the G dwarf problem if 90% of gas is blown out of the galaxy. This doesn't match observations for the Milky Way, but it does work for dwarf irregulars.

- Accreting Box Model

For large spirals, gas only accounts for a small amount of the total mass. Some infalling gas immediately turns into stars.

$$\frac{d}{dt} M_{tot} \neq 0$$

but $dM_s + dM_g = 0$.

Working through this model, we find that:

$$M_s(Z < Z(T)) = -M_g \ln(1 - \frac{Z}{\rho})$$

Note the log (as opposed to exponential) dependence.

Choosing $Z = \frac{1}{3} Z_0$, and using $M_g = 0.1 M_s$, we find:

$$\boxed{\frac{M_s(Z < \frac{1}{3} Z_0)}{M_s} = 0.04}$$

This solves the G dwarf problem.

Lecture 17

- Galactic Rotation

If we consider the Local Standard of Rest (LSR) to be at -r along the x axis and the galaxy to be more or less contained in a circle of radius r centered on the origin, then we should find that for observations within the solar circle, quantrants I and II should be receding (positive v), and III and IV should be approaching (negative v).

$$v_r = (\Omega(R) - \Omega_0) R_0 \sin \ell$$

where Ω_0 is the angular velocity of the sun around the galaxy.

For a given line of sight, there is a maximum v_r. For other measured velocities, there is an ambiguity.

Lecture 18

Last time we derived differential rotation in the Milky Way:

$$\theta = \frac{R}{R_0}\left(\frac{V_r}{\sin\ell\cos b} + \theta_0\right)$$

where ℓ is the galactic longitude, b is the galactic latitude, and v_r is the projection of θ along the line of sight, minus the projection of θ_0. Recall that galactic longitude is measured relative to our reference at the edge of the galaxy, not as an angle around the galactic center. This equation assumes circular, cylindrical rotation around the galactic center.

When looking through the galactic plane, we should see gas interior to our orbit to have a positive doppler shift in lines on sight toward the direction of our orbit, and negative doppler shifts in the opposite direction of our orbit. Exterior to our orbit, the signs of both of these cases changes. According to our model of cylindrical motion, we should see only the thermal doppler broadening around the inherent wavelength of our spectral line if we look straight up in galactic longitude. As an aside, we know that the temperature of H in our galaxy cannot exceed 10,000 K, which corresponds to a line width of about 10 $\frac{km}{s}$. If we see wider line widths, they have to be a result of turbulent motion.

Anyway, when we look straight straight up (or straight down), we find that gas is falling towards us. Thus, we don't have purely cylindrical motion. Likewise, when we look at galactic longitude 180°, we see a line which is close to zero doppler shift, but still slightly negative. Thus, we don't have perfectly circular motion.

Things get worse when we look towards the galactic center. We see emission, but we also see absorption, and at many different doppler shifts. When we look slightly above and below the galactic center in latitude, much of this complexity disappears. Thus, this complexity is associated with the galactic center, and is the result of extremely non-circular orbits.

Since we know where the velocity curves peak along the line of sight (where R is a minimum), we can infer a radius and measure a maximum velocity. Thus, we can make our own velocity curves and find that the rotation curve of our galaxy is flat.

In these plots of velocity along the line-of-sight, we see features which indicate there are high-velocity clouds falling into the midplane, and that there is a warp to the midplane of the galaxy.

Lecture 19

When we make observations through the Milky Way with a radio telescope, we are trying to sample $T_a(\ell, b, v)$ and use that information to tell us how gas is moving. However, our radio telescope has a beam width, and our spectrometer has bins of a certain width, so we essentially pixel-ize this data at some $d\ell, db, dv$. Changing to cylindrical coordinates centered on the center of the galaxy, we can translate our pixels $d\ell, db$ into $d\phi, dR, dz$. Now we are in physical coordinates, and we can use the equation:

$$N(HI) = 1.823 \cdot 10^{18} T_a dv$$

Now $N(HI) = \int \rho(\phi, R, z)dR$, or we can flip around and consider columns in z. If we want to find the thickness of the column of hydrogen along the z axis, we can just compute the moment:

$$\langle z^2 \rangle = \int_{-\infty}^{\infty} z\rho dz$$

Using data gathered with these z measurements, we can analyze the warping modes of the plane of our galaxy. We find that the warping of our galaxy is dominated by m=0 (bowl shaped), m=1 (integral shaped), and m=2 (saddle shaped). Weinberg at UMass makes an argument that the warping is the gravitational effect of satellite galaxies (like the Large Magellenic Cloud) on the dark matter halo of our galaxy, where distortions in the dark matter halo are giving rise to coherent, low-order, modal structure in our disk.

On to a different topic: we can determine the velocity dispersion of gas in the plane of our disk in our neighborhood by looking at some direction where we shouldn't see any postive-shifted gas, and attributing any positive velocities we see to the tail of a gaussian velocity distribution. Fitting for this, we find a dispersion in our neighborhood of about 7 km/s.

Suppose we would like to solve for some of the rotation constants of the Milky Way in the neighborhood of our sun–i.e.:

$$\frac{v_r}{R_0 \sin \ell \cos b} = \Omega - \Omega_0$$

and we want v_r for small $R - R_0$. To do this, we'll Taylor expand R_0:

$$\frac{v_r}{R_0 \sin \ell \cos b} = \overbrace{\Omega - \Omega_0\Big|_{R_0}}^{0} + (R - R_0)\frac{d\Omega}{dR}\Big|_{R_0} + \overbrace{\frac{1}{2}(R - R_0)^2 \frac{d^2\Omega}{dR^2}\Big|_{R_0}}^{0}$$

Now $R - R_0 = -r \cos \ell$, so we have

$$\begin{aligned}
\frac{v_r}{\cos b} &= -r R_0 \sin \ell \cos \ell \frac{d\Omega_0}{dR_0}\Big|_{R_0} \\
&= \frac{-r R_0}{2} \sin 2\ell \frac{d\Omega_0}{dR_0}\Big|_{R_0} \\
&= \underbrace{-\frac{1}{2} R_0 \frac{d\Omega_0}{dR_0}\Big|_{R_0}}_{const=A} r \sin 2\ell \\
&= A r \sin 2\ell
\end{aligned}$$

Changing variables from Ω to Θ, we have:

$$\begin{aligned}
\frac{d\Omega_0}{dR_0}\Big|_{R_0} &= \frac{1}{R_0}\left(\frac{d\Theta_0}{dR_0}\Big|_{R_0} - \frac{\Theta_0}{R_0}\right) \\
A &= \frac{1}{2}\left(\frac{\Theta_0}{R_0} - \frac{d\Theta_0}{dR_0}\Big|_{R_0}\right)
\end{aligned}$$

This is called the Oort A constant. The first measurement of this constant confirmed Shapely's theory that we are not in the center of the galaxy.

Lecture 20

- Oort Constants

Our tangential velocity with respect to the galactic center can be expressed as:

$$v_T = A\cos 2\ell r + Br$$

where A is a measurement of shear (determined from radial velocities), and B is a measurement of the local curl (determined from tangential velocities).

$$A - B = \Omega_0$$

$$A + B = -\left(R_0 \frac{d\Omega}{dR}\bigg|_{R_0} + \Omega_0 \right) = -\frac{d\theta}{dR}\bigg|_{R_0}$$

Measurements show $A = 15\frac{km}{s\cdot kpc}$ and $B = -10\frac{km}{s\cdot kpc}$. This implies $\Omega_0 = 25\frac{km}{s\cdot kpc}$ and $A + B = 5\frac{km}{s\cdot kpc}$.

We are on the inside edge of a spiral arm. Gas falling into the arm provides a shear beyond normal, so our A is abnormally high.

There are additionally Oort Constants C and K which measure shear along the tangential axis, and the divergence of the local velocities.

- Measurements of Constants

  R_0 is determined from RR Lyrae stars, globular clusters, proper motion of masers in the galactic center, and from statistical parallax. Statistical parallax is done using VLBI to get the position of H_2O masers with extreme accuracy, and then watching the masers move. We find the sun is about 8 kpc from the galactic center.

  Ω_0 is determined from A, B measurements (as described before) and by measuring the proper motion of Sgr A^*. We find that $b \approx 0$, but ℓ changes, so we deduce that it's motion is our motion.

- Gas Flow

  CO measurements at the galactic anticenter show inflow at 2-3 km/s.

  This gas flow replentishes gas which is depleted from star formation.

  Star formation proceedes at 1-5 $\frac{M_\odot}{yr}$, from a reservoir of $10^9 M_\odot$, so star formation would end in 10^9 yrs without this inflow.

Lecture 21

We start with the (collisionless) Boltzmann equation:

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left(v_i \frac{\partial f}{\partial x_i} - \frac{\partial \phi}{\partial x_i} \frac{\partial f}{\partial v_i} \right) = \frac{df}{dt} = 0$$

which says that the change in the distribution of velocities in a volume which is comoving with the fluid ($\frac{df}{dt}$), is related to the change in distribution of a non-comoving element ($\frac{\partial f}{\partial t}$), plus a factor which represents a flux over the boundary of the comoving volume (both in position and velocity space). This equation is not equal to 0 when collisions become important.

We now begin taking moments of this equation.

$$0 = \int_0^\infty \frac{\partial f}{\partial t} d^3\bar{v} + \sum \int_0^\infty v_i \frac{\partial f_i}{\partial x_i} d^3\bar{v} - \sum \frac{\partial \phi}{\partial x_i} \int_0^\infty \frac{\partial f}{\partial v_i} d^3\bar{v}$$
$$= \frac{\partial \int}{\partial t}_0^\infty f d^3\bar{v} + \frac{\partial}{\partial x_i} \int_0^\infty f v_i d^3\bar{v}$$

where the last term of the first equation was eliminated using the divergence theorem: $\int_V \vec{\nabla} f d^3\bar{x} = \int_S f d^2\bar{s} = 0$. Now $n \equiv \int_0^\infty f d^3\bar{v}$ is the number density of stars, and $\langle v_i \rangle = \bar{v}_i = \frac{\int f v_i d^3 v}{\int f d^3 v}$, so we have our first Jean's equation:

$$\boxed{\frac{\partial n}{\partial t} + \frac{\partial (n\bar{v}_i)}{\partial x_i} = 0}$$

This equation is the star analog for the continuity equation in fluids:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho \bar{v}_i}{\partial x_i}$$

For stars, instead of conserving mass, we conserve number. Next, we use:

$$0 = \frac{\partial}{\partial t} \int_0^\infty f v_j d^3\bar{v} + \sum \frac{\partial}{\partial x_i} \int_0^\infty v_i v_j f d^3\bar{v} - \underbrace{\sum \frac{\partial \phi}{\partial x_i} \int v_j \frac{\partial f}{\partial v_i} d^3\bar{x}}_{\textit{integrate by parts}}$$

Physically, the last term is $\overline{v_i v_j}$, and represents the correlation between velocities in two different directions. Instead of going through the derivation, we will jump straight to the second Jean's equation:

$$\boxed{n\frac{\partial \bar{v}_j}{\partial t} + n\bar{v}_i \frac{\partial v_j}{\partial x_i} = -n\frac{\partial \phi}{\partial x_i} - \frac{\partial (n\sigma_{ij}^2)}{\partial x_i}}$$

where $\sigma_{ij}^2 \equiv \overline{v_i v_j} - \bar{v}_i \bar{v}_j$. This equation is the star-equivalent of the fluid momentum equation:

$$\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla)\bar{v} = -\nabla \phi - \frac{1}{\rho} \nabla p$$

In fluids, we have the an energy conservation equation. In the stellar dynamics case, this equation is not useful because the only way to change the energy of a galaxy is to hit it with another galaxy. The one exception to this is in globular clusters, where stars can get evaporated out of the group.

Now, unfortunately, we need to talk about these equations in cylindrical coordinates. The Jeans conservation equation becomes:

$$\vec{\nabla} \cdot \bar{F} \frac{1}{R} \frac{\partial}{\partial R} \left(R F_R + \frac{1}{R} \frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z} \right)$$

where $\bar{F}_i = n v_i$

$$\boxed{\frac{\partial \bar{v}_\phi}{\partial \phi} = 0}$$

$$\frac{\partial n}{\partial t} + \frac{1}{R} \frac{\partial (T n \bar{v}_R)}{\partial R} + \frac{\partial n \bar{v}_z}{\partial z} = 0$$

The second equation becomes:

$$\frac{\partial n\bar{v}_z}{\partial t} + \frac{\partial (n\overline{v_R v_z})}{\partial R} + \frac{\partial (n\bar{v}_z^2)}{\partial z} + \frac{n\overline{v_R v_z}}{R} + n\frac{\partial \phi}{\partial z} = 0$$

For a plane-parallel sheet in steady state, we know that $\frac{\partial}{\partial t} = 0$, $\frac{\partial}{\partial R} = 0$, and $\bar{v}_R = 0$, so in our second equation, that leaves us with:

$$\boxed{\frac{\partial (n\bar{v}_z^2)}{\partial z} + n\frac{\partial \phi}{\partial z} = 0}$$

Which is an equation which agrees nicely with observations of disk galaxies. Combining this with Poisson's equation:

$$\frac{\partial^2 \Phi}{\partial z^2} = 4\pi G \rho_{TOT}$$

we get

$$\frac{1}{n}\frac{\partial (n\bar{v}_z^2)}{\partial z} = -\frac{\partial \Phi}{\partial z} \equiv -g_z$$

which is an acceleration in the z direction. Thus:

$$\boxed{\frac{d}{dz}\left[\frac{1}{n}\frac{d(n\bar{v}_z^2)}{dz}\right] = -4\pi G \rho_{TOT}}$$

Thus, we can measure the velocity and number density in the z direction and we can get an estimate of the total mass density in the midplane of a galaxy. Typically, this is done using K giants because they are old (and therefore, more representative of the total population). We make the assumption here that all stars have the same dynamical temperature ($\bar{v}_z^2$). It turns out that this assumption breaks down, as it was discovered that there are "old thin disk" and "thick disk" stellar populations.

If we integrate the above equation, we can get an equation for the total density along a column $\Sigma_{TOT} = \int_0^\infty \rho(z)dz$. This determination was used to show there was no excess dark matter associated with the disk compared to the halo.

Graphing $g(z)$, we find it grows linearly with z up to 300 pc, with $g(z)_{300} = 5 \cdot 10^{-9} cms^{-2}$. Because $g(z)$ grows linearly, we can infer that $\rho(z)$ is relatively constant near the midplane. We measure $\boxed{\rho_0 = 0.1 M_\odot pc^{-2}}$ and $\Sigma_{disk}(R_0) = 48 \pm 9 M_\odot pc^{-2}$

Lecture 22

We now ask the question: what is the form of g_z for intermediate z (300 pc to 30-50 kpc)? We know that:

$$\nabla^2 \phi = -\vec{\nabla} \cdot \vec{g} = 4\pi G \rho_{TOT}$$

Then using Stokes' Law:

$$\int_V \vec{\nabla} \cdot \vec{g} dV = \int_S \vec{g} dA = 4\pi G \int \rho_{TOT} dA$$

Because the mass at intermediate z is dominated by stars (dark matter is still spherically symmetric around our system), we can treat all the mass as being concentrated in an infinite sheet at the midplane. Thus, this problem becomes identical to the electrostatics problem of the flux from an infinite charged sheet, and we can invoke Gauss's Law and the symmetry of the problem to say that for a box drawn to include a section of the midplane, with faces parallel to the sheet, we have that:

$$\vec{g} dA = 2g_z A$$

where A is the area of a face parallel to the sheet. Thus, we have

$$\boxed{\vec{g} = -2\pi G \Sigma_* \hat{z}}$$

Where Σ_* is the surface density of stars.

For large z,

$$\nabla_r^2 \Phi = 4\pi G \rho$$
$$\frac{1}{r^2} \frac{dr^2}{dr} \frac{d\phi}{dr} = 4\pi G \rho_{TOT}$$
$$-g_r = \frac{Gm}{r}$$

Thus we have an overall picture of g_z vs. z which increases linearly for small z, is flat for a long time, and then decreases as $\frac{1}{r}$ for large z. In the flat regime we can calculate:

$$g_z = 2\pi G \Sigma_* = \frac{2\pi(6.67 \cdot 10^{-8}) \cdot 70 \cdot 2 \cdot 10^{33}}{(3 \cdot 10^{18})^2} = 6.2 \cdot 10^{-9}$$

Also, $g_z \equiv \ddot{z}$, so we can say $\ddot{z} = 4\pi G \rho_0 z$, which has solution $z = Ae^{i\omega t}$ for $\omega^2 = 4\pi G \rho_0$. Thus, for a star close to the midplane (like our sun):

$$\omega = \sqrt{\frac{4\pi(7 \cdot 10^{-8}) \cdot 0.1 \cdot 2 \cdot 10^{33}}{(3 \cdot 10^{18})^3}} = 2 \cdot 10^{-15}$$

We can work out that the period of this oscillation is about 83 million years. It has been suggested that this period for midplane crossings corresponds to the period for mass extinctions on earth by comet/asteroid collisions.

For a midrange star, $\ddot{z} = 2\pi G \Sigma_*$, so $z = \pi G \Sigma_* t^2$. This says that the period depends on where the star starts. We'll say 1 kpc, which gives us the solution:

$$\frac{1}{4} period = \left(\frac{z}{\pi G \Sigma_*}\right)^{\frac{1}{2}} = 1.06 \cdot 10^9 years$$

So we've worked out something of the motion of an individual star, but now we'd like to work out the distribution of stars in z for a population. We'll assume that we have an infinite plane-$\|$ sheet.

$$\frac{d}{dz}\left[\frac{1}{\rho}\frac{d(\rho \bar{v}_z^2)}{dz}\right] = -4\pi G \rho_{TOT}$$
$$\frac{d}{dz}\left[\frac{1}{\rho_*}\frac{d(\rho_* \bar{v}_z^2)}{dz}\right] = -4\pi G \rho_*$$

We'll now change variables $\rho \rightarrow \Lambda$, and $z \rightarrow \zeta$:

$$z = \zeta \left(\frac{\bar{v}_z^2}{4\pi G \rho_0}\right)^{\frac{1}{2}}$$
$$\rho = \rho_0 \Lambda(\zeta)$$

and differentiating, we have:

$$dz = d\zeta \left(\frac{\bar{v}_z^2}{4\pi G \rho_0} \right)^{\frac{1}{2}}$$

$$\frac{d}{d\zeta} \frac{1}{\Lambda(\zeta)} \frac{d\Lambda(\zeta)}{d\zeta} = -\Lambda(\zeta)$$

And this has the ultimate solution:

$$\boxed{\Lambda(\zeta) = sech^2 \left(\frac{\zeta}{\sqrt{2}} \right)}$$

where, you recall, $sech(u) \equiv \frac{2}{e^u + e^{-u}}$. From this messy equation we can infer the units of the following:

$$\frac{v_z^2}{4\pi G \rho_0} \rightarrow cm^2$$

So we'll define

$$h_* \equiv \left(\frac{v_z^2}{2\pi G \rho_0} \right)^{\frac{1}{2}}$$

as our scale height, giving us that

$$\boxed{\rho = \rho_0 sech^2 \left(\frac{z}{h_*} \right)}$$

To find what z is when $\rho_* = \frac{1}{2}\rho_0$, we'll define $u \equiv \frac{z}{h_*}$ so that:

$$\frac{1}{2}\rho_0 = \rho_0 \left(\frac{2}{e^u + e^{-u}} \right)^2$$

We find that this has solution $u = 0.8814$, so:

$$z_{\frac{1}{2}} = 0.8814 \left(\frac{\bar{v}_z^2}{2\pi G \rho_0} \right)^{\frac{1}{2}}$$

From Binney & Merrifield, we have that in the solar vicinity $v_z = 18 \frac{km}{s}$, and $\rho_0 = 0.1 \frac{M_\odot}{pc^3}$, so we have for a thin disk:

$$z_{\frac{1}{2}} = 304pc$$

For the thick disk, $v_z = 39 \frac{km}{s}$, so $z_{\frac{1}{2}} = 1.4kpc$. From observations, we find that $z_{\frac{1}{2}}$ does not change with radius. Note that for small heights, $sech^2 \left(\frac{z}{h_*} \right)$ is close to $e^{-\left(\frac{z}{h_*} \right)^2}$. Similarly, in the asymptotic limit of $z \rightarrow \infty$, we find that

$$\Sigma_* \left(\frac{z}{h_*} \right) = \left(\frac{2\rho_0 v_z^2}{\pi G} \right)^{\frac{1}{2}} = 69 M_\odot pc^{-2}$$

which agrees well with observation ($71 \pm 6 M_\odot pc^{-2}$).

Lecture 23

Today we'll be talking about the Jean's instability, ram-pressure stripping, and tidal forces (all topics pertaining to interstellar gas).

Jean's Instability

We start with steady-state fluid equations for mass and momentum conservation:

$$\frac{d\rho}{dt} = 0$$
$$\rho\frac{dv}{dt} = -\nabla p - \rho\nabla\phi$$

We will then perturb $\rho, v, p,$ and ϕ, keeping only first-order perturbations, and write down the differences between this perturbation and the steady-state solution:

$$\frac{\partial\rho_1}{\partial t} + \rho_0\vec{\nabla}\cdot\vec{v}_1 + v_0\vec{\nabla}\rho_1 = 0$$
$$\frac{\partial v_1}{\partial t} = (\vec{v}_0\cdot\vec{\nabla})\vec{v}_1 + (\vec{v}_1\cdot\vec{\nabla})v_0 = \frac{\rho_1}{\rho_0^2}\nabla p_0 - \frac{1}{\rho_0}\nabla p_1 - \nabla\phi_1$$

We will then make the barotropic assumption that $p = p(\rho)$ only. Then we can say $p_1 = c^2\rho_1$, where c is the (isothermal) sound speed of the gas. Using this in our previous equations, we have:

$$\frac{\partial v_1}{\partial t} = (\vec{v}_0\cdot\vec{\nabla})\vec{v}_1 + (\vec{v}_1\cdot\nabla)\vec{v}_0 = -c^2\nabla\left(\frac{\rho_1}{\rho_0}\right) - \nabla\phi_1$$

This equation, along with $p_1 = c^2\rho_1$ and the two equations above, we have 4 equations. Assuming that $\vec{v}_0 = 0$, $\rho_0 = const$, and $\phi_0 = 0$, our equations become:

$$\frac{\partial\rho_1}{\partial t} + \rho_0\vec{\nabla}\cdot\vec{v}_1 = 0$$
$$\frac{\partial v_1}{\partial t} = -c^2\nabla\frac{\rho_1}{\rho_0} - \nabla\phi_1$$
$$\nabla^2\phi_1 = 4\pi G\rho_1$$

Taking $\frac{\partial}{\partial t}$ of the first equation and combining it with ∇ of the second equation, we have:

$$\underbrace{\frac{\partial^2\rho_1}{\partial t^2} - c^2\nabla^2\phi_1}_{wave\ equation} - 4\pi G\rho_1\rho_0 = 0$$

For the wave equation portion, we'll assume a solution $\rho = Ce^{i(\vec{k}\cdot\vec{x}-\omega t)}$, giving us a dispersion relation:

$$\boxed{\omega^2 - c^2k^2 - 4\pi G\rho_0}$$

For large wavenumber ($k \gg \frac{4\pi G\rho_0}{c^2}$), we have $\omega^2 = c^2k^2$. As wavenumbers decrease, we arrive at an instability when $k_J^2 = \frac{4\pi G\rho_0}{c^2}$. This is the Jeans Wavenumber. As a wavelength this reads:

$$\boxed{\lambda_J^2 = \frac{\pi c^2}{G\rho_0}}$$

This equation is simply a relation between the sound crossing time ($\frac{c}{\lambda}$) and the free-fall time $\sqrt{G\rho_0}$. Let us remind ourselves that this relation is for collapsing gas. The Jeans criterion for instability in star systems is:

$$\lambda_J = \left(\frac{G\rho_0}{\pi\sigma_*^2}\right)$$

where σ_* is the velocity dispersion of stars.

We may go on to ask what the Jeans mass M_J at recombination might be:

$$M_J = \frac{4}{3}\pi\rho(\frac{1}{2}\lambda_J)^3 = \frac{1}{6}\pi\rho_0\left(\frac{\pi c^2}{G\rho_0}\right)^{\frac{3}{2}}$$

At recombination, $\rho_0 \sim 10^{-23}$. The temperature over this time period goes from $T \sim 10^4 \to 10^2 K$. Then using $\frac{1}{2}mc^2 = \frac{3}{2}kT$, and assuming $T \sim 1000K$, (and possibly remembering that gas at 1000K has a velocity dispersion of $3\frac{km}{s}$) we have:

$$c^2 = \frac{3kT}{1.4m_H} = 10\left(\frac{km}{s}\right)^2$$

This gives us a Jeans mass of $M_J \sim 3 \cdot 10^6 M_\odot$. This is about the mass of a dwarf galaxy of a globular cluster, suggesting that these objects were the first to form in our universe.

We can apply this work on Jeans criteria to stars and star formation. Let's try to figure out how hot the gas that forms a $100M_\odot$ star can be, given observed densities of $n = 10^3 cm^{-3}$:

$$M_J = \frac{\pi}{6}\rho_0\left(\frac{3\pi kT}{G\rho_0 1.4m_{H_2}}\right)^{\frac{3}{2}} = 100M_\odot$$

This gives us a temperature of $T \sim 32K$. Note that we use the mass of M_{H_2} because almost all of the hydrogen in these cool clouds is molecular. Observationally, we find temperatures around $10K$ and densities around $n_0 \sim 10^5$, giving us $M_J \sim 6.4M_\odot$. Naively, we'd assume this should be the mass of most of the stars we see. We know, however, that most of the mass of stars is in $\frac{1}{2}M_\odot$ stars. To explain this, we need for star formation to proceed inefficiently, with much of the mass being ejected during collapse.

For rotating disks, we have a Jeans equation:

$$\omega^2 = \kappa^2 - 2\pi G\Sigma\,|k| + k^2c^2$$

where $\kappa = R\frac{d\Omega^2}{dR} + 4\Omega^2$ is the epicyclic frequency. For gas in this system, we have the criterion for instability being $Q < 1$, where Q is:

$$Q \equiv \frac{c\kappa}{\pi G\Sigma_g}$$

where Σ_g is the surface density of gas. For stars we have:

$$Q \equiv \frac{\sigma\kappa}{3.36G\Sigma_*}$$

For the Milky Way, we find $\Sigma_* \sim 50\frac{M_\odot}{pc^2}$, $\sigma \sim 30km\ s^{-1}$, and $\kappa \sim 36km\ s^{-1}kpc^{-1}$, giving us $Q = 1.7$, which suggests that the galaxy is overall stable, but may become unstable in spiral arms. This work is recent by a guy named Toomre.

Ram Pressure

The idea here is that hot, static, x-ray emitting gas in the galaxy clusters imparts pressure on the galaxies moving around inside. The ram pressure that the galaxy imparts on the gas is ρv^2, where v is the velocity of the galaxy relative to the gas. This ram pressure can cause stripping of gas off of the galaxy if it exceeds the hydrostatic pressure of gas in the galaxy:

$$P_{hydro} = \alpha G \Sigma_* \Sigma_g$$
$$\rho_{IGM} > \frac{\alpha G \Sigma_* \Sigma_g}{v^2}$$

Using that $\Sigma_* \sim 50 \frac{M_\odot}{pc^2}$, and $\Sigma_g \sim 7 \frac{M_\odot}{pc^2}$ we find that this effect is present in the Milky Way.

Lecture 24

Tidal Forces

In addition to the normal tidal forces which act to stretch an orbitting body along the radial action of gravity, **tidal shear** results when an object (like a globular cluster) differentially orbits around a massive body (that is, the near edge is moving in orbit more quickly than the far edge). The degree of shear depends on the relation between the differential rotation and the self-gravity of the system. In practice, we are in a rotating frame (our sun) which is also rotating, so we either need to find an inertial frame of reference or account for the Coriolis forces.

In a rotating frame, the force per unit mass is:

$$\ddot{R} = -\frac{\theta^2}{R} + R\Omega^2$$

where R is the distance from the galactic center.

$$\frac{dF_R}{dR} r = \left(\frac{\theta}{R^2} - \frac{2\theta}{R}\frac{d\theta}{dR} + \Omega^2 \right) r$$
$$= 2r\Omega_0 \left(\Omega_0 - \left.\frac{d\theta}{dR}\right|_{R_0} \right)$$

where $\Omega_0 = A - B$, where A and B are the Oort constants. We may then write the condition for stability as:

$$\boxed{\frac{dF_R}{dR} r \geq 4r(A - B)}$$

where r is the distance across the orbitting body. Futhermore, the acceleration from self-gravity on the cluster is:

$$g = \frac{F}{m_*} = \frac{GM}{r^2} = \frac{G\frac{4}{3}\pi\rho_{cc}r^3}{r^2}$$

Thus, the condition for stability is:

$$\boxed{\rho_{cc} \geq \frac{3}{\pi G} A(A - B)}$$

If a cluster were hypothetically in the solar neighborhood, this would come out to be:

$$\rho_{cc} \geq \frac{3 \cdot 235 \cdot 15 \cdot 25}{10^6} = 8.4 \cdot 10^{-2} \frac{M_\odot}{pc^3}$$

In fact, as has been stated many times, the density in the solar neighborhood is $0.1 \frac{M_\odot}{pc^3}$, so such objects in our neighborhood would be tidally stable.

For another case, suppose we are in an inertial frame (we are the galaxy). In this case, we have no Coriolis term:

$$\frac{F_R}{m} = -\frac{\theta_c^2}{R}$$
$$\frac{dF_R}{dR} = \frac{\theta_c^2}{R^2} - 2\frac{\theta_c}{R}\frac{d\theta_c}{dR}$$

where θ_c is the circular angular velocity of our galaxy around the center of mass. Using $g = \frac{4}{3}\pi\rho_r r^2 \frac{G}{r^2}$, we have the condition for stability is:

$$\frac{dF_R}{dR} r < \frac{GM}{r^2}$$

$$\boxed{\left(\frac{\theta_c^2}{R^2} - 2\frac{\theta_c}{R}\frac{d\theta_c}{dR}\right) r < \frac{4}{3}\pi G \rho_{cc}}$$

If we assume we have a flat rotation curve, then $\frac{d\theta_c}{dR} = 0$, so we have

$$\boxed{\rho_{cc} = \frac{3}{4\pi G}\frac{\theta_c^2}{R^2} = \frac{3}{4\pi G}\Omega^2}$$

For a dwarf galaxy at 50 kpc, this comes to:

$$\rho_{gal} = \frac{3 \cdot 235 \cdot 220^2}{4\pi(50 \cdot 10^3)^2} = 1.1 \cdot 10^{-3} \frac{M_\odot}{pc^3}$$

We can estimate the density of a dwarf galaxy: it has 10^6 stars and has a diameter of about $1kpc$, so we have $\rho_{cc} \sim 10^{-3} \frac{M_\odot}{pc^3}$, so this galaxy would be on the border of stable. Anything closer (like the Sagittarius Dwarf) will get totally ruined. By the way, the "235" in the above equations is G in units of km, s, and pc's.

Lecture 25

Spiral Structure (Density Waves)

This material is covered in Chapter 6 of the new Binney and Tremaine, and in Chapter 3 of Binney and Tremaine for a discussion of epicyclic frequencies.

We define the **pitch angle** to be the angle between the tangent of a circular orbit and the spiral at that point. Most grand-design spirals (spirals which are symmetric, usually have 2 arms, and stretch from the center to the edge of a galaxy) have pitch angles between 5° and 20°, with $\sim 10°$ being typical. There is a good discussion of this in 1964 ApJ (Lin & Shu).

Spiral arms last for about a Hubble time. At the distance of our sun from the center of the galaxy, this is something like 50 orbits. Using that the circular velocity is given by:

$$v_c = \left(\frac{GM}{R}\right)^{\frac{1}{2}}$$

we can compute the total angular momentum of a galaxy:

$$\begin{aligned} L_{tot} &= 2\pi \int R^2 v_c(R)\Sigma(R)dR \\ &= 2\pi (GM)^{\frac{1}{2}} \int R^{\frac{3}{2}}\Sigma(R)dR \end{aligned}$$

The energy of rotation is:

$$\begin{aligned} E_{tot} &= 2\pi \int (\frac{1}{2}v_c^2(R) + \Phi(R))\Sigma(R)dR \\ &= -\pi GM \int \Sigma(R)dR \end{aligned}$$

The change in these quantities as a result of a small mass δm being transported outward (from $R_i \rightarrow R_f$), is:

$$\begin{aligned} \Delta L &= \delta m \left[(GMR_f)^{\frac{1}{2}} - (GMR_i)^{\frac{1}{2}}\right] \rightarrow diverges \\ \Delta E &= -\delta m \left(\frac{GM}{R_f} - \frac{GM}{R_i}\right) \rightarrow bounded \end{aligned}$$

Thus, with a small piece of mass, we can transport any amount of angular momentum out of the system, but not an arbitrary amount of energy.

Looking at stellar structure in a galaxy (M51) with a grand design spiral, we see that spiral structure is very faint, but present, in the stars which are responsible for the bulk of the gravitational potential in the plane of the galaxy. Looking at gas emission, we find that the inner edges of spiral arms have molecular gas, the middle of the arms consists of atomic gas, and the outer edges have ionized gas. If we guess that molecular gas collapses into stars which ionize the rest of the gas, this suggests that gas enters the insides of spiral arms and flows outward across the arm. The width of the ionized portion of the spiral structure is determined by how long molecular clouds last and are able to produce massive O9 stars to create HII regions. Once these molecular clouds stop producing massive stars (which don't live very long), HII regions rapidly recombine to atomic gas, and as this atomic gas gets pulled and compressed into the next spiral arm, the likewise increasing density of dust acts as an effective coolant, counteracting heating from compression and allowing atomic gas to cool to a level that allows molecular clouds to form.

From what we observe, spiral structure is nearly stationary. This means that spiral structure must undergo solid-body rotation. Thus, as we go farther out, the rotational speed of the spiral structure must increase, and at the **corotation resonance point**, will achieve the same orbital speed as the gas and stars.

To from grand design spirals, there must be a non-axisymmetric perturbation, usually caused by a companion galaxy or a bar, and there must be a linearly increasing rotation curve.

Lecture 26

- Winding Problem

  Our galaxy differentially rotates. If we assume that spiral arms hang around for about a Hubble time ($10^{10} yrs$), and if spiral arms move with the rotation of the galaxy, then we'd expect arms to have a characteristic angle of $0.15°$. However, we measure an angle of $\sim 5°$.

  Either spiral arms are short-lived (in which case we should see many fewer spiral galaxies with clear arms–but we see about $\frac{1}{2}$), or spirals are density waves kicked up by a precessing bar.

- Precessing Bar

  Considering the form of a gravitational potential in a rotating reference frame, there is a potential for an oscillating potential with:

  $$\kappa^2 = R\frac{d(\Omega^2)}{dR} + 4\Omega^2$$

  or written in terms of the Oort constants:

  $$\kappa^2 = -4B\Omega$$

  This results in a pattern which precesses with angular velocity:

  $$\Omega_p = \Omega - \frac{n\kappa}{m}$$

  where n, m are integers representing different harmonics. In practice, the lower harmonics are excited with greatest amplitude.

Lecture 27

More on Epicyclic Frequencies

We began yesterday by deriving our epicyclic frequency:

$$\kappa^2 = R\frac{d\Omega^2}{dR} + 4\Omega^2$$

and is a measure of the commensality of orbits in the rotating frame. That is, it measures the oscillation of non-circular orbits around the circular orbital frequency. For our sun, this looks like:

$$\kappa^2 = -4B\Omega$$

where the Oort constants are: $A = 14.5\frac{km}{s \cdot kpc}$, $B = -12\frac{km}{s \cdot kpc}$, and $\Omega = 27\frac{km}{s \cdot kpc}$. Using that the sun is 8 kpc we can get the velocity of our sun in its orbit. Plugging the Oort constants into our equation for the epicyclic frequency, we get that $\kappa = 38\frac{km}{s \cdot kpc}$. We also showed last time that for keplarian orbits, $\kappa = \Omega$. In this case, our star makes 1 orbit on it's epicycle per revolution around the galactic center. Drawing this out (and remembering that the epicyclic orbits rotate around with the star as it goes around the galaxy), we find that a circular orbit becomes an ellipse. This is an $m = 1$ perturbation of an orbit, using the perturbation formula:

$$\boxed{\Omega_p = \Omega - \frac{n\kappa}{m}}$$

For a solid body, we derived last time that $\kappa = 2\Omega$. Tracing this out, we get a "bar-like" orbit. This is an $m = 2$ perturbation. Now if we consider many of these orbits nested inside of one another,

In general, we can characterize a potential by plotting Ω vs. R. For solid-body rotation, we have Ω constant vs. R. Next, let's say that $\Omega = \frac{V}{R}$, and V is constant. Then, we have $\Omega \propto \frac{1}{R}$. Now if we draw the epicyclic perturbations $\kappa = 2\Omega$ and $\kappa = \sqrt{2}\Omega$, they parallel Ω, and converge to 0. We can then consider rotation curves $\Omega \pm \frac{\kappa}{2}$ (which are resonances in our orbital system), and these rotation curves are *flat*.

Now if we have a constant perturbation (or pattern) Ω_p superimposed on a rotation curve, we inevitably have a point where $\Omega = \Omega_p$, where we have **corotation**. This, obviously, creates a resonance because the normal rotation feeds energy into the perturbation. Similarly, we have a resonance (called the **Outer Lindblad resonance**) at $\Omega + \frac{\kappa}{2}$, where rotation feeds energy in phase with the orbit (and this continues for $\frac{\kappa}{3}$ and so on, but to a lesser extent.

Let's take an example: if we have a 2 arm spiral that moves round with pattern speed Ω_p, then a star at radius where $\Omega = \Omega_p$, we have energy fed into the pattern. Farther out (or closer in), it may not be that $\Omega = \Omega_p$, but rather, it matches with a combination of the orbital frequency and the epicyclic frequency. This will feed energy into the epicyclic orbit, reinforcing the pattern.

There are also **Inner Lindblad resonances** (there can be 2 of them if the curve $\Omega - \frac{\kappa}{2}$ turns over). Further inside, we have V going linearly with R, which is solid-body rotation. In this regime, all perturbed orbits rotate around with one another, giving us a **bar**. This can last out to the corotation point. Outward of this point, these resonances wind back (the phases of the epicycles changes), giving us spirals.

Let's try to calculate the winding time of a pattern, using $\phi_p = \phi_0$ at $t = 0$, where ϕ_p is the angle of the major axis of the pattern. The difference in rotation speeds of different points on the pattern causes a drift:

$$\phi_p(t) = [\Omega - \frac{\kappa}{2} - \Omega_p]t$$

Earlier, when we did the winding problem, we had $\cot\alpha = \left|R\frac{d\phi}{dt}\right|$, where α was the pitch angle. Plugging this in for our galaxy, we get $\alpha = 1.4°$ (as opposed to .25°, which we calculated earlier) after $10^{10}yrs$. This is a big improvment. We measure angles from $5 \to 15°$, and if we allow for $3 \cdot 10^9 yrs$ to pass (instead of the Hubble time), we get winding angles this large. Just judging by the rarity of grand-design spirals, we have reason to believe they might be temporal phenomena, so this estimate is not unreasonable.

The rest of what we didn't cover today is in Chapter 6 of Binney & Tremaine.

Lecture 28

The Galactic Center

We cannot observe the galactic center in optical because there are ~ 30 magnitudes of extinction. The first observations were made in radio by Carl Jansky. We can estimate the mass of the bulge in our galaxy by either doing velocity dispersion measurements or taking mass-to-light ratios using CO emission from red giants, or by doing dust measurements and using dust-to-gas-to-star ratios. The mass of the disk of our galaxy is about $10^{11}M_\odot$, and the mass of the bulge is about $10^{10}M_\odot$. Since the bulge is about 1 kpc in radius, we can estimate that the density of the bulge is about $3M_\odot$ per pc^3 (compared to $0.1M_\odot pc^{-3}$ for the disk). We can figure the gravitation potential in the bulge:

$$\nabla^2\phi = 4\pi G(\rho_* + \rho_g)$$
$$\frac{\nabla P_g}{\rho} = -\nabla\phi$$
$$P_g = \rho_g v_g^2$$

The second equation is the equation for hydrostatic equilibrium, and we are going to make the assumption that $\rho_g \ll \rho_*$, because the gas will always be confined to the plane of the disk. We then find:

$$\frac{d(\rho_g v_g)^2}{dz} = \rho_g \frac{d\phi}{dz}$$
$$v_g^2 \frac{d\rho_g}{\rho_g} = v_g^2 d\ln\rho_g = \frac{d\phi}{dz}$$
$$\frac{\rho_g(z)}{\rho_g(0)} = exp\left(-\frac{1}{v^2}\int_0^z g_z dz\right)$$

If we assume ρ_* is constant then we have:

$$\nabla^2\rho = 4\pi G\rho_*$$
$$\nabla\phi = 4\pi G\rho_* z$$
$$\rho_g(z) = exp\left(-\frac{1}{v_g^2}2\pi G\rho_*(0)z^2\right)$$
$$= e^{-\frac{z^2}{h_*^2}}$$

where $h_*^2 \equiv \frac{v_g^2}{2\pi G\rho_*(0)}$ is the scale height of gas in the bulge. We can also get the pressure as:

$$\boxed{P = 0.84\Sigma_g\Sigma_*^{\frac{1}{2}}\frac{v_g}{h_*^{\frac{1}{2}}}}$$

This tells us that the pressure in the galactic center is about $10^{2.7}$ times more than in the solar neighborhood.

Looking at velocity vs. longitude in the galactic center, we see a lot of gas at forbidden velocities (receding when it should be approaching and visa versa from circular orbits). If we have a bar in the center of our galaxy, then we have stars in noncircular orbits because of epicyclic motion. Gas cannot have epicyclic motion because it is collisional. Instead, gas has elogated orbits resulting from the disturbed potential of the stars going around in the bar. Inside of the inner Lindblad resonance (about 20 pc from the center in the MW), gas can still have circular orbits, as it can outside the outer Lindblad resonance. These regions are called X2 and X1 orbits, respectively.