UC Berkeley, Astro 202, Astrophysical Gas Dynamics

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Lecture 1

- $\overleftrightarrow{\sigma}$ is the **stress tensor** and each σ_{ij} term represents the i^{th} component of a pressure across a plane surface normal to j. The stress tensor is symmetrical, or else surface torques would impart infinite angular momentum as $\delta V \to 0$. The trace of σ is related to the **hydrostatic pressure** on a fluid element:

$$\frac{1}{3} trace(\sigma) = -p$$

- **Volume forces** F are defined to be per-fluid-element accelerations, and **body forces** F_{body} represent the total force on a fluid element, such that:

$$F_{body} = \rho F$$

The force a fluid element exerts via pressure over its boundary is given by:

$$-\int_S p n_A \; dA = -\int_V \nabla p \; dV$$

Thus, the resultant body force is ∇p. The **condition for hydrostatic equilibrium** is then:

$$\boxed{\rho F = \nabla p}$$

For the case of a self-gravitating fluid in hydrostatic equilibrium, $F = \nabla \phi$, where ϕ, the graviatational potential, is given by:

$$\nabla^2 \phi = 4\pi G \rho$$

Thus, we may eliminate ϕ from the equation of hydrostatic equilibrium, yielding:

$$\vec{\nabla} \cdot \left(\frac{\nabla p}{\rho} \right) = -4\pi G \rho$$

Lecture 2

- The **equation of continuity** states that the rate of change of density at a given location must depend solely on the flow of mass over a bounding surface (assuming mass is not created or annihilated in the fluid). This equation reads:

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho u) = 0}$$

- It is often useful to have a derivative which follows fluid elements along their direction of flow. For this, we define a **material derivative**:

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u \cdot \vec{\nabla} \cdot$$

  Using his definition, we can see how a fluid element can be accelerated along a streamline, even if the fluid velocity at a particular point in space never changes. A **streamline** is the path a fluid element takes in a steady flow (time-invariant velocity field).

- Using our material derivative, we may now describe the volume force (acceleration) of a fluid element as $\frac{Du}{Dt}$. Thus, the $F = ma$ of fluid dynamics, called the **Euler Equation** is:

$$\boxed{\rho \frac{Du}{Dt} = -\nabla p + \rho F}$$

  We may expand the $u \cdot \nabla u$ in $\frac{Du}{Dt}$ using a vector identity worth remembering:

$$u \cdot \nabla u = \vec{\nabla} \times u \times u + \frac{1}{2} \nabla u^2$$

  We then define the **vorticity** ω of a fluid to be the curl of the velocity field at a given point:

$$\omega \equiv \vec{\nabla} \times u$$

- An **incompressible fluid** makes the assumption of constant pressure, and is equivalent to:

$$\vec{\nabla} \cdot u = 0$$

- We may write the Euler equation in terms of vorticity. Let us do this for a self-gravitating fluid ($F = -\nabla \phi$):

$$\frac{\partial u}{\partial t} + w \times u + \frac{1}{2} \nabla u^2 = -\frac{1}{\rho} \nabla p - \nabla \phi$$

  In steady flow, $\frac{\partial u}{\partial t} = 0$. Taking the dot product of u with this equation and noting that ρ is constant along streamlines ($\frac{D\rho}{Dt} = 0$ from continuity and incompressibility), we get:

$$u \cdot \nabla \left(\frac{1}{2} u^2 + \frac{p}{\rho} + \phi \right) = 0$$

  Since in steady flow, $\frac{D}{Dt} = u \cdot \nabla$, we have:

$$\boxed{\frac{1}{2} u^2 + \frac{p}{\rho} + \phi = constant}$$

  for incompressible flow along streamlines. This is **Bernoulli's theorem**.

- So far we've neglected the internal friction (viscosity) of a fluid. We define μ, the **coefficient of shear viscosity**, to be the drag between two flows per bordering area, per velocity gradient:

$$\frac{F}{A} = \mu \frac{du_x}{dz}$$

In a 3 dimensional fluid, flow in any direction (say $\hat{z}$) drags against fluid in the $\hat{x}$ and $\hat{y}$ directions. We keep track of all of these components in a **viscous stress tensor** $\overleftrightarrow{\pi}$, where:

$$\pi_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \mu' (\vec{\nabla} \cdot u) \delta_{ij}$$

The additional $\mu' (\vec{\nabla} \cdot u) \delta_{ij}$ term disappears for incompressible flow, but comes into play for compression or expansion. μ' is called the **coefficient of bulk viscosity**, and for a monatomic gas, $\mu' = -2\mu$. The body force resulting from viscosity is:

$$\boxed{F_{visc} = \vec{\nabla} \cdot \overleftrightarrow{\pi}}$$

For an incompressible fluid, this is simply $F_{visc} = \mu \nabla^2 u$. This approximation is frequently made, but beware that it makes it impossible to treat the attenuation of sound waves.

- Viscosity enters the equation of motion as a force term in the Euler equation. Taking the curl of both sides of this equation yields:

$$\frac{\partial \omega}{\partial t} + \vec{\nabla} \times (\omega \times u) = \frac{\mu}{\rho} \nabla^2 \omega$$

We then define the **coefficient of kinematic viscosity** to be:

$$\nu = \frac{\mu}{\rho}$$

This is the coefficient typically referred to in literature. Note that the expression of the Euler equation above (neglecting the $\vec{\nabla} \times (\omega \times u)$ term) is a diffusion equation, and suggests that $t_{diff} \sim \frac{\ell^2}{\nu}$, where ℓ is the characteristic length of a region of vorticity.

- Consider the flow of a fluid around an object (or equivalently, an object through a fluid). Say flow proceeds around this object as a speed U, and that the object has a characteristic diameter D. It turns out that all of the factors necessary to characterize flow in this system come lumped together in one number called the **Reynolds Number**:

$$\boxed{R = \frac{UD}{\nu}}$$

We derive this from the Euler equation, scaling distances to D and velocities to U. Flows with the same Reynolds number will look similar *on the characteristic scales of U and D*. This is only true if compressibility can be neglected (otherwise, this only holds for flows with the same ratio to the sound speed).

- The drag on an object in a flow depends in a complicated manner on R. However for $R \ll 1$, it is generally the case that the drag coefficient (the force of drag per ram pressure $\frac{1}{2}\rho U^2 D\ell$, where $D\ell$ is the cross-section of the object) is given by $C_{drag} = \frac{24}{R}$.

- As the Reynolds number of a flow increases, the flow pattern becomes increasingly chaotic. Initially, eddies form behind the object. As R increases, these eddies get shed downstream with increasing frequency. Ultimately, these merge into a quasi-uniform, turbulent wake behind the object.

Lecture 3

- All of the fluid equations we've written down may be derived from a kinetic theory of fluids. If $f(\vec{x},\vec{v},t)$ tells us the number of particles at any position, velocity, and time, we may compute the macroscopic value of some quantity $Q(\vec{x},t)$ as an average over/moment of f:

$$\langle Q(\vec{x},t)\rangle = \frac{1}{n}\int d^3\vec{v}\ Q(\vec{x},\vec{v},t)\ f(\vec{x},\vec{v},t)$$

For example, the bulk velocity $\vec{u}$ is:

$$\vec{u} = \frac{1}{n}\int d^3\vec{v}\ \vec{v}\ f(\vec{x},\vec{v},t)$$

and the thermal velocity dispersion is:

$$\langle w^2\rangle = \frac{1}{n}\int d^3\vec{v}\ (\vec{v}-\vec{u})^2\ f(\vec{x},\vec{v},t)$$

where $\frac{1}{2}m\langle w^2\rangle = \frac{3}{2}kT$. The energy density is:

$$\rho\epsilon = \frac{1}{n}\int d^3\vec{v}\ \frac{1}{2}m\vec{v}^2\ f(\vec{x},\vec{v},t)$$

where ϵ is the **entropy**.

- In a kinetic theory of fluids, $f(\vec{x},\vec{v},t)$ evolves because of collisions between particles. Assuming collisions are nearly instantaneous, we may model them as sources/sinks which remove particles from one point in phase space and add them to another. This yields the **Boltzmann Equation**:

$$\boxed{\frac{\partial f}{\partial t} + \dot{x}_i\frac{\partial f}{\partial x_i} + \dot{v}_i\frac{\partial f}{\partial v_i} = \left(\frac{\partial f}{\partial t}\right)_C}$$

where

$$\left(\frac{\partial f}{\partial t}\right)_C \equiv \int \vec{v}_{rel} f_1 \sigma(\vec{v}_{rel},\Omega) f_2 d\Omega d^2v$$

is a **collision operator** acting on f, and $\vec{v}_{rel}$ is the relative velocity between particle 1 and particle 2. For quantities which are conserved by collisions, we may set $\left(\frac{\partial f}{\partial t}\right)_C = 0$. We then take moments of this equation by multiplying it by a quantity Q and integrating over $\vec{v}$.

Lecture 4

- Using the technique above (and some tricky algebra), we may derive an equation for the **conservation of momentum**:

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x_i}(\rho u_i u_k + p\delta_{ik} - \pi_{ik}) + \rho\frac{\partial \phi}{\partial x_i} = 0$$

  where π_{ik} is a component of the viscous stress tensor $\overleftrightarrow{\pi}$, and ϕ is a potential field (say, from gravity). This is equivalent to the Euler equation for force.

- We may follow a similar procedure to derive an equation for the **conservation of energy**:

$$\boxed{\frac{\partial E}{\partial t} + \vec{\nabla}\cdot\rho\left(\frac{1}{2}u^2 + h\right)\cdot u = u\rho\vec{\nabla}\cdot\phi}$$

  where $E = \rho\left(\frac{1}{2}u^2 + \epsilon\right)$ is the total energy of a fluid element, ϵ is the entropy (energy per density), h is the **enthalpy** (energy per volume).